Studies in Fuzziness and Soft Computing

Pankaj Gupta
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Suresh Chandra

Fuzzy Portfolio Optimization

Advances in Hybrid Multi-criteria Methodologies

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To my parents Smt. Vimal Gupta and
(Late) Shri Govind Ram - Pankaj Gupta

To my parents Smt. Santosh and
Shri Shish Ram - Mukesh Kumar Mehlawat

To (Late) Professor Hideo Tanaka
- Masahiro Inuiguchi

To my parents (Late) Smt. Shivpyari Devi and
(Late) Shri Uma Nath Srivastava - Suresh Chandra

Foreword

Portfolio optimization in general, and asset portfolio optimization in particular, are concerned with an optimization of the capital proportion of the assets held that results in the largest return possible with the lowest risk encountered. The concepts of portfolio optimization have played significant role in the development and understanding of various facets of investment decision-making. The major breakthrough in the field came in 1952 when Harry Markowitz published his modern portfolio theory commonly known as mean-variance portfolio theory being widely used by investors to construct portfolios that are based on the trade-off between risk and return under various economic conditions. The mean-variance framework considers that returns on assets follow a normal distribution whereas risk is articulated with the use of standard deviation. To alleviate the limitations of the mean-variance framework, various alternative risk measures have been proposed in the literature along with consideration of criteria other than risk and return. Modern optimization techniques have been instrumental in solving large scale portfolio optimization problems in an efficient manner.

The environment in which the investment decisions are made is inherently uncertain. Furthermore such decisions are human-centric and dwell upon invoking and processing linguistic information. Moreover, keeping pace with the emerging discourse on corporate conduct, functioning of the financial markets and economic development, the portfolio optimization approaches have also been extended by engaging with and effectively quantifying psychological preferences and the biases of the investor to generate more balanced portfolios that are based on trade-off between financial criteria and non-financial criteria.

This research monograph authored by the leaders in the area, Professors Pankaj Gupta, Mukesh Kumar Mehlawat, Masahiro Inuiguchi, and Suresh Chandra offers a well-structured, comprehensive, fully updated, and lucidly written treatise on various optimization techniques used in the investment decision-making. The book covers a broad spectrum of vital problems of portfolio analysis. It starts with a focused introduction to the fundamental problem and then elaborates on the advanced optimization techniques which help the investors in obtaining well-diversified portfolios engaging both financial and non-financial criteria. Chapter after chapter, the authors navigate the reader through a vast and exciting teritory of advanced decision-making in portfolio optimization.

One of the indisputable features of the monograph is a well-delineated linkage between the concepts, theory, algorithms, and practice. The carefully selected numeric studies help establish and highlight these crucial relationships.

Considering the rapidly growing complexity of real-world problems, the portfolio optimization models have to become reflective of this by embracing more advanced methodology and invoking more elaborate algorithmic means. As a matter of fact, the book is a tangible testiomony to this visible trend. Fuzzy sets come as a conceptual vehicle - they are capable of capturing the granular nature of the problems. Uncertainty conveyed by linguistic terms is conveniently formalized in the language of fuzzy sets and possibility theory. The pertinent optimization framework embraces a variety of tools originating from possibilistic programming, credibility theory, fuzzy multicriteria optimization, evolutionary optimization and support vector machines.

In summary, the book is a *must* for everyone interested in pursuing advanced research and/or engaged in practical issues of portfolio optimization.

Witold Pedrycz
University of Alberta
Edmonton, Canada

Preface

"A good portfolio is more than a long list of good stocks and bonds. It is a balanced whole, providing the investor with protections and opportunities with respect to a wide range of contingencies."

Harry Markowitz (1959)

Portfolio optimization has certainly come of the age of the twin parameters of risk and return in view of heterogeneity of investor characteristics and expectations as well as sheer variety of financial products. Moreover, asset return and risk themselves defy precise measurement on account of both the ambiguity of their expressions as well as the uncertainties of the environment in which they occur. Additionally, the growing instances of corporate scams and scandals have made it incumbent upon the investors to consider the psychological preferences, the quality of governance of corporations and ethicality of their conduct. This monograph is written to provide a systematic framework to understand portfolio optimization problem from the real-world perspective using multiple criteria decision making models.

How does one account for the myriad of variables that come into interplay for portfolio optimization: the subjective preferences of the investor, demographic, sociological and psychological determinants of the subjective preferences, estimates of the asset returns, risk, liquidity from past data, expert-advice, asset fundamentals such as company earnings, management and governance and so on? How does one account for ambiguities, uncertainties and vagueness associated with operational definitions of many of these variables?

The objective of this monograph is to traverse the transition of portfolio optimization right from the evolution of the research area and basic models to its extension into the domain of fuzzy set theory, multiple criteria decision making and hybrid approaches. Our main emphasis in this monograph is to provide an overview of the discipline of portfolio optimization from return-risk-liquidity perspectives. Real-world uncertainties of portfolio optimization are best described in fuzzy terms. It is, therefore, natural to elaborate portfolio optimization under fuzziness. Although, portfolio optimization considers return and risk as the two fundamental criteria, it is true that all the relevant information for portfolio optimization can not be captured in terms of return and risk only. The other criteria might be of equal, if not greater, importance for portfolio optimization. For example, in the recent past, the investors are becoming conscious of the desirability of ethical evaluation of the assets and the consideration of the psychological preferences and the biases as well. Hence, the research in the area must take cognizance of these developments to construct models that accord due consideration to ethical and suitability criteria besides the financial criteria. This monograph presents a comprehensive discussion of the hybrid multi-criteria framework for portfolio optimization in fuzzy environment. In addition, this monograph includes a new frame work for behavioral models of portfolio optimization by discussing suitability and ethicality considerations along with financial optimality.

The monograph is structured as follows. A brief overview of portfolio optimization is presented in Chapter 1 in which we discuss the classical mean-variance model of portfolio optimization developed by Markowitz and various extensions of the mean-variance model by considering alternative measures of risk. The main emphasis in Chapters 2-5 is to introduce various important concepts used in real-world applications of portfolio optimization such as interval numbers, fuzzy decision theory, possibility theory and credibility theory. The portfolio optimization models using these concepts have also been discussed in detail. In Chapter 2, we first review interval numbers and interval arithmetic and then, portfolio optimization models using interval coefficients in respect of model parameters are presented. Chapter 3 serves as the foundation of the fuzzy portfolio optimization models presented in this monograph. It contains a brief overview of fuzzy decision theory and presents a fuzzy framework of the mean-variance portfolio optimization model using max-min approach. Chapter 4 is devoted to a thorough study of the foundations of possibility theory and the portfolio optimization problem with fuzzy coefficients. Chapter 5 contains a detailed discussion of the credibility theory and a credibilistic framework for the portfolio optimization problem. Chapters 6-9 are mainly focused on introducing behavioral considerations in multi-criteria portfolio selection and also presents systematic framework to incorporate the subjective preferences of the investor. In Chapter 6, we discuss portfolio optimization models that incorporate individual investor attitudes towards portfolio risk, namely, aggressive (weak risk aversion attitude) and conservative (strong risk aversion attitude). Chapter 7 contains a fuzzy framework of portfolio selection by simultaneous consideration of suitability and optimality. Our focus is to attain the convergence of suitability and optimality in portfolio selection by evolving a typology of investors, categorizing the financial assets into different clusters and by performing suitability evaluation of the assets based on investor preferences. In Chapter 8, we continue the study of suitability and optimality in portfolio selection by applications of advanced multi-criteria approaches. Chapter 9 concerns the discussion of another important issue in portfolio selection that is based on socially responsible investment. The portfolio optimization framework presented in this chapter involves ethical and financial evaluation of the assets using investor preferences. The concluding chapter of this monograph is on application of support vector machines and real-coded genetic algorithms in portfolio optimization. The portfolio optimization models presented in each chapter are validated using real data set extracted from National Stock Exchange, Mumbai, India.

The potential readers of this monograph include academicians, researchers and practitioners in the realms of portfolio optimization. This monograph is most suitable to those who are presently working in the area of portfolio optimization or wish to understand the basics of portfolio optimization to do research in this area. Further, it is suitable as a text book to several graduate and masters programs that have specialized course on portfolio optimization. Although, authors have put their best to make the presentation error free, some errors may still remain and we hold ourself responsible for that and request that the errors, if any, be intimated by emailing at pgupta@or.du.ac.in (e-mail address of Pankaj Gupta).

We would first and foremost like to thank Professor Janusz Kacprzyk for accepting our proposal to publish the monograph in the Springer series on Studies in Fuzziness and Soft Computing. We would like to thank the editors and publishers of the journals 'Information Sciences', 'Fuzzy Sets and Systems', 'Knowledge-Based Systems' 'Expert Systems with Applications', 'Journal of Global Optimization' and 'International Journal of Information Technology and Decision Making' for publishing our research work in the area of fuzzy portfolio optimization which constitute the core of this monograph.

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(Delhi, India), Pankaj Gupta
(Delhi, India), Mukesh Kumar Mehlawat
(Osaka, Japan), Masahiro Inuiguchi
(Delhi, India) Suresh Chandra

Contents

Chapter 1
Portfolio Optimization: An Overview

Abstract. In this chapter, we present a brief overview of portfolio optimization. First, we discuss the classical mean-variance model of portfolio optimization developed by Markowitz. We then discuss the various extensions of the Markowitz's model by considering alternative measures of risk, namely, semivariance, absolute deviation and semi-absolute deviation.

1.1 Mean-Variance Model

'Do not put all your eggs in one basket' is an age old wisdom capturing the fundamental idea underlying portfolio optimization. The 'wisdom' essentially lies in the return-risk characteristics of the various assets. Clearly, 'more' assets may not necessarily be 'good' if all the assets exhibit the same return-risk characteristics. A 'good' portfolio is the one that gives higher return for a given level of risk or the one that gives lower risk for a given level of return. Thus, a 'good' portfolio would comprise assets that are different rather than similar in terms of these characteristics. Operationalization of the age old wisdom necessitated mathematical modeling for portfolio optimization. The mathematical problem of portfolio optimization can be formulated in many ways but the principal problems can be summarized as follows:

(i) Minimize risk for a specified expected return
(ii) Maximize the expected return for a specified risk
(iii) Minimize the risk and maximize the expected return using a specified risk aversion factor
(iv) Minimize the risk regardless of the expected return
(v) Maximize the expected return regardless of the risk

The solutions of the first three problems are called mean-variance efficient solutions. The fourth problem gives minimum variance solutions which are desirable for conservative investors. It is also used for comparison and benchmarking of other portfolios. The fifth problem gives the upper bound of the expected return which can be attained; this is also useful for comparisons.

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Harry Markowitz made the major breakthrough in 1952 with the publication on portfolio selection theory [90]. The Markowitz's theory, popularly referred to as modern portfolio theory, provided an answer to the fundamental question: How should an investor allocate capital among the possible investment choices? Markowitz suggested that it is impossible to derive all possible conclusions concerning portfolios. A portfolio analysis must be based on some criteria which serve as a guide to the important and unimportant, the relevant and irrelevant. The proper choice of criteria depends on the nature of the investor. For each type of investor the details of the portfolio analysis must be suitably selected. However, the two criteria that are common to all investors are expected (mean) return and variance of return (risk). Markowitz assumed that 'beliefs' or projections about assets follow the same probability rules that random variables obey. From this assumption, it follows that (i) the expected return on the portfolio is a weighted average of the expected returns on individual assets, and (ii) the variance of return on the portfolio is a particular function of the variances of and the covariances between assets, and their weights in the portfolio. Hence, investors must consider risk and return together and determine the allocation of capital among investment alternatives on the basis of the trade-off between them.

Further, Markowitz suggested that portfolio selection should be based on reasonable beliefs about future rather than past performances *per se*. Choices based on past performances alone assume, in effect, that average returns of the past are good estimates of the 'likely' return in the future; and variability of return in the past is a good measure of the uncertainty of return in the future.

In what follows next, we present the mathematical formulation of the mean-variance model proposed by Markowitz [90]. Let R_i be a random variable representing the rate of return (per period) of the i-th asset ($i = 1, 2, \ldots, n$). Also, let x_i be the proportion of the total funds invested in the i-th asset.

Definition 1.1 (Asset return). The asset return is expressed as the rate of return which is defined during a given period as

$$\Big((\text{closing price for the current period}) - (\text{closing price for the previous period}) + (\text{dividend(s) for the current period})\Big) \Big/ (\text{closing price for the previous period})$$

Note that the period of return may be a day or a week or a month or a year.

In particular, for the i-th asset the realization r_{it} of the random variable R_i during period t ($t = 1, 2, \ldots, T$) is defined as

$$r_{it} = \frac{(p_{it}) - (p_{it-1}) + (d_{it})}{(p_{it-1})},$$

where p_{it} is the closing price of the i-th asset during the period t, p_{it-1} is the closing price during the period $t-1$, d_{it} is the dividend of the i-th asset during the period t.

For example, the yearly return of an asset in 2012 is calculated as follows:

$$\frac{(\text{closing price, 2012}) - (\text{closing price, 2011}) + (\text{dividend(s), 2012})}{(\text{closing price, 2011})}$$

The investor would have gained or lost the above amount if he/she invested Rs. 1.00 at the end of 2011, collected the dividend(s) declared in 2012 and sold at the closing price of 2012. A loss is represented by a negative return. Suppose if the closing price of 2011 was 36, that of 2012 was 40 and 2 is the dividend declared during 2012, then the return in 2012 would be

$$\frac{(40) - (36) + (2)}{(36)} = 0.1667$$

or a gain of 16.67% per rupee invested.

Definition 1.2 (Portfolio). A portfolio is a collection of two or more assets represented by an ordered n-tuple $\Theta = (x_1, x_2, \ldots, x_n)$, where x_i is the proportion of the total funds invested in the i-th asset.

The expected return (per period) of the investment (portfolio) is given by

$$r(x_1, x_2, \ldots, x_n) = E\left[\sum_{i=1}^{n} R_i x_i\right] = \sum_{i=1}^{n} E[R_i] x_i = \sum_{i=1}^{n} r_i x_i \,,$$

where $E[\cdot]$ represents the expected value of the random variable in the bracket and $r_i = E[R_i]$. The expected value of the random variable can also be approximated by the average derived from the past data, i.e.,

$$r_i = E[R_i] = \frac{1}{T} \sum_{t=1}^{T} r_{it} \,. \tag{1.1}$$

An investor prefers to have portfolio return ($r(x_1, x_2, \ldots, x_n)$) as large as possible. Further, an investor would also prefer that the portfolio return should have minimum possible dispersion/variability. Markowitz suggested that variance which measures the dispersion from expected return can be used to quantify portfolio risk. The variance of the i-th asset denoted by σ_i^2 is expressed as

$$\sigma_i^2 = v(R_i) = E\left[(R_i - E[R_i])^2\right] = E\left[(R_i - r_i)^2\right].$$

He also suggested standard deviation as another measure of dispersion. The standard deviation of the i-th asset is expressed as

$$\sigma_i = \sqrt{v(R_i)} = \sqrt{E\left[(R_i - E[R_i])^2\right]} = \sqrt{E\left[(R_i - r_i)^2\right]}\,.$$

Note that variance of return on a portfolio is not determined solely by the variances of the individual asset returns. It also depends on the covariance between return on assets. The covariance σ_{ij} between asset returns R_i and R_j is expressed as

$$\sigma_{ij} = E\left[(R_i - E[R_i])(R_j - E[R_j])\right].$$

Using past data, the covariance σ_{ij} can be approximated as follows:

$$\sigma_{ij} = \frac{1}{T}\sum_{t=1}^{T}(r_{it} - r_i)(r_{jt} - r_j)\,. \tag{1.2}$$

Further, σ_{ij} may be expressed in terms of the correlation coefficient (ρ_{ij}) as follows:

$$\sigma_{ij} = \rho_{ij}\ \sigma_i\ \sigma_j\,.$$

The variance of return on a portfolio is thus obtained as

$$\begin{aligned}
v(x_1, x_2, \ldots, x_n) &= E\left[\left(\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right)^2\right] \\
&= E\left[\left(\sum_{i=1}^{n} R_i x_i - \sum_{i=1}^{n} r_i x_i\right)^2\right] = E\left[\left(\sum_{i=1}^{n}(R_i - r_i)x_i\right)^2\right] \\
&= E\left[\sum_{i=1}^{n} x_i^2 (R_i - r_i)^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_i x_j (R_i - r_i)(R_j - r_j)\right] \\
&= \sum_{i=1}^{n} x_i^2 E\left[(R_i - r_i)^2\right] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_i x_j E\left[(R_i - r_i)(R_j - r_j)\right] \\
&= \sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} x_i x_j \sigma_{ij}\,.
\end{aligned}$$

If we use the fact that $v(R_i) = \sigma_{ii}$, then we have

$$v(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j \sigma_{ij}\,.$$

Remark 1.1. *It is important to point out that while the standard deviation is stated in terms of the rate of returns, the variance is stated in terms of* *the rate of returns squared. As it is more natural to interpret rate of returns rather than rate of returns squared, risk is usually measured with standard deviation of returns. However, for calculation purposes, it is usually more convenient to use the variance rather than the standard deviation. Either risk measure is appropriate because the standard deviation is merely a simple mathematical transformation of the variance.*

Definition 1.3 (Short selling). It refers to a situation where an investor actually does not own an asset but he/she establishes a market position by selling the asset in anticipation that the price of that asset will fall. In such situations, the investor is said to have taken a short position. Mathematically, this situation can be explained by taking the number of assets owned by the investor as negative.

In the portfolio analysis, we exclude the negative values of x_i (i.e., short selling is not allowed); therefore, $x_i \geq 0$ for all i ($i = 1, 2, \ldots, n$). Also, since x_i is the proportion of the total funds invested therefore $\sum_{i=1}^{n} x_i = 1$.

We now present two different formulations of the Markowitz's mean-variance model based on the following assumptions derived from the above discussion:

(i) The prices of all assets at any time are strictly positive.
(ii) The rate of return R_i ($i = 1, 2, \ldots, n$) is a random variable taking finitely many values.
(iii) An investor can own a fraction of an asset. This assumption is known as divisibility.
(iv) An asset can be bought or sold on demand in any quantity at the market price. This assumption is known as liquidity.
(v) There are no brokerage/transaction costs.
(vi) Short selling of an asset is not permitted.

Case 1: The portfolio optimization model for minimizing variance and constraining the expected portfolio return is formulated as follows:

$$\textbf{P(1.1)} \quad \min \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^{n} r_i x_i = r_0 \,, \tag{1.3}$$

$$\sum_{i=1}^{n} x_i = 1 \,, \tag{1.4}$$

$$x_i \geq 0 \,, \quad i = 1, 2, \ldots, n \,, \tag{1.5}$$

where r_0 is the portfolio return desired by the investor. The objective function of the problem P(1.1) minimizes the portfolio risk (variance). Constraint (1.3) ensures that the expected portfolio return at the end of a holding period must be equal to target value r_0 desired by the investor. Constraint (1.4) represents capital budget constraint on the assets and constraint (1.5) ensures no short selling of assets. The problem P(1.1) is a quadratic programming problem. By varying the desired level of return (r_0) and repeatedly solving the quadratic programming problem, the minimum variance portfolio for each value of r_0 can be obtained.

Remark 1.2. *Regarding the choice of r_0, the investor should not aspire for a very high return which is unrealistic to be achieved from the assets under consideration. In other words, if the investor desires for a very high return then the problem P(1.1) may become infeasible. The achievable portfolio return value, i.e., r_0 always lies between r_{min} and r_{max}. The r_{min} is the value of r_0 corresponding to a portfolio with minimum variance. That is, it is the return corresponding to the portfolio obtained by solving the problem P(1.1) excluding the constraint (1.3). The r_{max} is the maximum feasible r_0, i.e., it is the maximum mean return among the mean returns of assets.*

Case 2: The portfolio optimization model for maximizing the expected portfolio return and constraining variance of the portfolio is formulated as follows:

$$\mathbf{P(1.2)} \quad \max \sum_{i=1}^{n} r_i x_i$$

$$\text{subject to}$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j = v_0 \tag{1.6}$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,,$$

where v_0 is the portfolio risk (variance) that the investor is willing to take. Note that unlike r_0 in problem P(1.1), it is not an easy task to find the range for v_0 in which it lies. However, it is possible to find the upper limit of the range for v_0, i.e., v_{max}. It is the value v_0 of a portfolio with maximum portfolio return. In other words, it is the variance corresponding to the portfolio obtained by solving the problem P(1.2) excluding the constraint (1.6). We can generate different portfolios by considering the values $v_0 \leq v_{max}$. Further, the value v_0 should not be too small; otherwise, the problem P(1.2) may become infeasible.

Definition 1.4 (Efficient portfolio). A feasible portfolio x is called efficient if it has the maximal expected return among all portfolios with the same variance or alternatively, if it has a minimum variance among all portfolios that have the same expected return.

In order to exemplify, let us consider two portfolios A and B. According to the definition of efficient portfolio, A will be preferred to B if:

$$E(A) > E(B), \quad v(A) = v(B),$$

or

$$E(A) = E(B), \quad v(A) < v(B),$$

where $E(A)$ = expected return of portfolio A; $E(B)$ = expected return of portfolio B; $v(A)$ = variance of portfolio A and $v(B)$ = variance of portfolio B.

The expected return and variance of every investment opportunity can thus be calculated and plotted as a single point on the mean-variance diagram, as shown in Fig. 1.1.

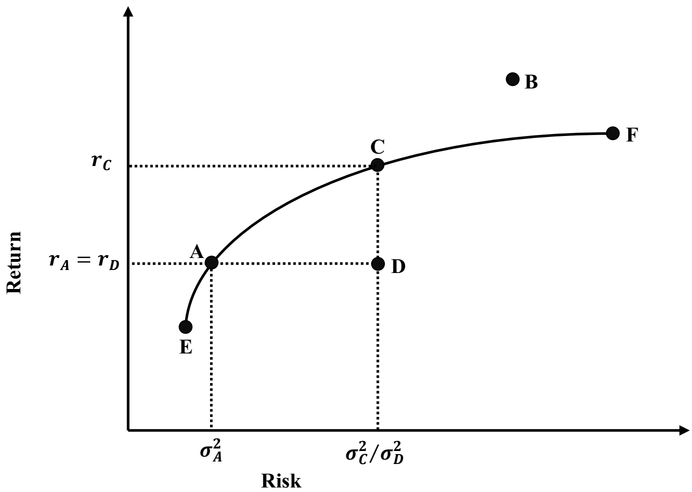

Fig. 1.1 Efficient frontier in portfolio analysis

It is important to point out that all the points on or below the curve EF represent portfolio combinations that are possible. Point D represents a portfolio of return r_D and risk σ_D^2. All points above the curve EF are combinations of risk and returns that do not exist. Point B would, therefore, represent risk and return that cannot be possibly obtained using any combination of assets. Also, point E represents a portfolio with 100% investment in minimum variance portfolio, alternatively, a point of minimum variance. On the other hand, point F represents a portfolio with 100% investment in single asset having maximum expected return.

Definition 1.5 (Efficient frontier). The curve obtained corresponding to efficient portfolios of the optimization problem P(1.1) or P(1.2) is called an efficient frontier.

Based on the definition of efficient frontier, the curve EF is also called efficient frontier because all points below the curve are dominated by any point on the curve. For example, suppose an investor is willing to take a portfolio risk σ_D^2. Now, the investor can obtain a portfolio return r_D with portfolio D or move to point C on the frontier and receive a higher portfolio return r_C in comparison to portfolio D. Therefore, portfolio C dominates portfolio D because it would give higher return for the same level of risk.

A similar argument could be made in terms of risk. If the investor wishes to achieve a return r_A, he/she will select portfolio A over D, because portfolio A represents the same return at a smaller level of risk, i.e., $\sigma_A^2 < \sigma_D^2$. Therefore, portfolio D is not efficient but portfolios A and C are.

Remark 1.3. *The value r_{min} of the portfolio corresponding to problem P(1.1)/P(1.2) is the return from the portfolio with minimum variance (see portfolio E in Fig. 1.1). On the other hand, the value r_{max} of the portfolio is the return from the portfolio by investing the entire capital in the asset having maximum return among assets under consideration (see portfolio F in Fig. 1.1).*

- **Numerical illustration**

The portfolio selection models P(1.1) and/or P(1.2) are validated using a data set extracted from National Stock Exchange (NSE), Mumbai, India which is the 9th largest stock exchange in the world by market capitalization. It is also the largest stock exchange in the world in terms of the number of trades in equities. Further, it is world's second fastest growing stock exchange too. Domestically, it is the largest by daily turnover and number of trades, for both equities and derivative trading.

We have randomly selected 10 assets listed on NSE to form a population from which we attempt to construct portfolios. Our sample data include daily closing prices of the 10 assets covering the period from April 1, 2007 to March 31, 2008. We use the average returns based on the average of the averages, that is, the average monthly returns to obtain the expected return, variance and covariance for the selected assets. Table 1.1 provides the historical returns for the entire period of the study in respect of each asset.

Table 1.1 Returns of the assets for the period April 1, 2007 to March 31, 2008

	Monthly returns											
	1	2	3	4	5	6	7	8	9	10	11	12
A B B Ltd. (ABL)	0.07200	0.32032	0.29710	0.23600	-0.05161	0.50633	-0.02516	0.90484	0.03214	0.45968	0.22700	-0.87871
Alfa Laval (India) Ltd. (ALL)	-0.14433	0.19032	0.75032	0.03433	-0.33581	0.24700	0.49968	0.27032	-0.32786	0.31968	0.19933	-0.50903
Bajaj Hindusthan Ltd. (BHL)	0.08667	1.05613	0.05516	0.27567	-0.21839	0.49233	1.11516	0.57613	0.17143	0.92258	0.22367	-0.67903
Crompton Greaves Ltd. (CGL)	-0.18567	0.76774	0.16194	0.48633	-0.20710	0.47833	0.25710	0.59484	-0.02321	0.55387	0.07333	-0.11871
Hero Honda Motors Ltd. (HHM)	0.18233	0.33000	0.13677	0.46533	-0.12774	0.56067	0.10839	0.00000	0.14321	0.00968	-0.15767	-0.27258
Hindustan Construction Co. Ltd. (HCC)	-0.15700	0.61226	1.23548	0.56067	-0.71065	0.97333	0.32839	0.61581	0.03286	0.49935	-0.03733	-0.59452
Kotak Mahindra Bank Ltd. (KMB)	0.18567	0.27806	0.55097	0.02733	-0.46613	0.73333	0.20581	0.17065	-0.05286	0.66710	0.37300	-0.08355
Mahindra & Mahindra Ltd. (MML)	0.37533	0.65903	0.19290	0.16533	-0.15226	0.80867	0.39097	0.29000	0.19750	0.21839	0.03100	-0.06548
Siemens Ltd. (SIL)	-0.10467	0.25516	0.31161	0.43333	-0.31710	1.10400	0.37194	0.73097	0.03321	0.75903	0.09467	-0.44903
Unitech Ltd. (UNL)	0.26367	0.41581	0.24484	0.12967	-0.08290	0.54000	0.93258	0.61871	0.22750	0.68968	0.65433	0.65258

First, we use equation (1.1) to calculate the expected return of the asset ABL based on the data given in Table 1.1 as follows:

$$
\begin{aligned}
r_1 &= E[R_1] = \frac{1}{12}\sum_{t=1}^{12} r_{1t} \\
&= \Big(0.07200 + 0.32032 + 0.29710 + 0.23600 + (-0.05161) + 0.50633 \\
&\quad + (-0.02516) + 0.90484 + 0.03214 + 0.45968 + 0.22700 + (-0.87871)\Big)\Big/12 \\
&= 0.17499.
\end{aligned}
$$

On the same lines, we calculate the expected returns of the remaining nine assets. The expected returns of all the assets are provided in Table 1.2.

Table 1.2 Input data corresponding to expected return

Company	Return
ABL	0.17499
ALL	0.09950
BHL	0.33979
CGL	0.23657
HHM	0.11487
HCC	0.27989
KMB	0.21578
MML	0.25928
SIL	0.26859
UNL	0.44054

Now, we use equation (1.2) to calculate the variance of the asset ABL based on the data given in Tables 1.1-1.2 as follows:

$$
\begin{aligned}
\sigma_{11} &= \frac{1}{12}\sum_{t=1}^{12}(r_{1t} - r_1)(r_{1t} - r_1) \\
&= \Big((0.07200 - 0.17499)^2 + (0.32032 - 0.17499)^2 + (0.29710 - 0.17499)^2 \\
&\quad + (0.23600 - 0.17499)^2 + (-0.05161 - 0.17499)^2 + (0.50633 - 0.17499)^2 \\
&\quad + (-0.02516 - 0.17499)^2 + (0.90484 - 0.17499)^2 + (0.03214 - 0.17499)^2 \\
&\quad + (0.45968 - 0.17499)^2 + (0.22700 - 0.17499)^2 + (-0.87871 - 0.17499)^2\Big)\Big/12 \\
&= 0.16656.
\end{aligned}
$$

Table 1.3 Input data corresponding to variance and covariance

Company	ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL
ABL	0.16656	0.08967	0.12861	0.08818	0.04405	0.15995	0.06892	0.05260	0.14154	0.00366
ALL	0.08967	0.12562	0.11421	0.06378	0.02641	0.16708	0.08474	0.04011	0.10279	0.03682
BHL	0.12861	0.11421	0.25614	0.12394	0.05332	0.16096	0.08375	0.08739	0.14855	0.06449
CGL	0.08818	0.06378	0.12394	0.10279	0.04060	0.13204	0.05656	0.05247	0.10984	0.02941
HHM	0.04405	0.02641	0.05332	0.04060	0.05677	0.08892	0.03101	0.04920	0.06321	-0.01296
HCC	0.15995	0.16708	0.16096	0.13204	0.08892	0.32041	0.14144	0.09670	0.20118	0.02667
KMB	0.06892	0.08474	0.08375	0.05656	0.03101	0.14144	0.10648	0.05322	0.10631	0.04434
MML	0.05260	0.04011	0.08739	0.05247	0.04920	0.09670	0.05322	0.06992	0.07734	0.01978
SIL	0.14154	0.10279	0.14855	0.10984	0.06321	0.20118	0.10631	0.07734	0.18959	0.04272
UNL	0.00366	0.03682	0.06449	0.02941	-0.01296	0.02667	0.04434	0.01978	0.04272	0.07689

Also, we use equation (1.2) to calculate the covariance between assets ABL and ALL based on the data given in Tables 1.1-1.2 as follows:

$$\sigma_{12} = \frac{1}{12}\sum_{t=1}^{12}(r_{1t} - r_1)(r_{2t} - r_2)$$
$$= \Big((0.07200 - 0.17499)(-0.14433 - 0.09950) + (0.32032 - 0.17499)$$
$$(0.19032 - 0.09950) + (0.29710 - 0.17499)(0.75032 - 0.09950)$$
$$+ (0.23600 - 0.17499)(0.03433 - 0.09950) + (-0.05161 - 0.17499)$$
$$(-0.33581 - 0.09950) + (0.50633 - 0.17499)(0.24700 - 0.09950)$$
$$+ (-0.02516 - 0.17499)(0.49968 - 0.09950) + (0.90484 - 0.17499)$$
$$(0.27032 - 0.09950) + (0.03214 - 0.17499)(-0.32768 - 0.09950)$$
$$+ (0.45968 - 0.17499)(0.31968 - 0.09950) + (0.22700 - 0.17499)$$
$$(0.19933 - 0.09950) + (-0.87871 - 0.17499)(-0.50903 - 0.09950)\Big)\Big/12$$
$$= 0.08967$$

On the same lines, we calculate the remaining variance and covariances values. The data corresponding to variance and covariance are provided in Table 1.3.

To find an optimal asset allocation, we use the model P(1.1). For the purpose, we first formulate the model P(1.1) using the input data from Tables 1.2-1.3 as follows:

$$\begin{aligned}
\min\ & 0.16656x_1x_1 + 0.12562x_2x_2 + 0.25614x_3x_3 + 0.10279x_4x_4 \\
& +0.05677x_5x_5 + 0.32041x_6x_6 + 0.10648x_7x_7 + 0.06992x_8x_8 \\
& +0.18959x_9x_9 + 0.07689x_{10}x_{10} + 0.17934x_1x_2 + 0.25722x_1x_3 \\
& +0.17363x_1x_4 + 0.08810x_1x_5 + 0.31991x_1x_6 + 0.13785x_1x_7 \\
& +0.10520x_1x_8 + 0.28308x_1x_9 + 0.00732x_1x_{10} + 0.22842x_2x_3 \\
& +0.12756x_2x_4 + 0.05282x_2x_5 + 0.33415x_2x_6 + 0.16949x_2x_7 \\
& +0.08022x_2x_8 + 0.20558x_2x_9 + 0.07364x_2x_{10} + 0.24788x_3x_4 \\
& +0.10665x_3x_5 + 0.32192x_3x_6 + 0.16751x_3x_7 + 0.17479x_3x_8 \\
& +0.29710x_3x_9 + 0.12898x_3x_{10} + 0.08120x_4x_5 + 0.26409x_4x_6 \\
& +0.11311x_4x_7 + 0.10494x_4x_8 + 0.21969x_4x_9 + 0.05881x_4x_{10} \\
& +0.17784x_5x_6 + 0.06202x_5x_7 + 0.09840x_5x_8 + 0.12642x_5x_9 \\
& -0.02591x_5x_{10} + 0.28287x_6x_7 + 0.19339x_6x_8 + 0.40237x_6x_9 \\
& 0.05333x_6x_{10} + 0.10643x_7x_8 + 0.21261x_7x_9 + 0.08868x_7x_{10} \\
& +0.15467x_8x_9 + 0.03957x_8x_{10} + 0.08543x_9x_{10}
\end{aligned}$$

subject to

$$0.17499x_1 + 0.09950x_2 + 0.33979x_3 + 0.23657x_4 + 0.11487x_5$$
$$+0.27989x_6 + 0.21578x_7 + 0.25928x_8 + 0.26859x_9 + 0.44054x_{10} = r_0,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1,$$
$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,.$$

In order to solve the above model, we need to decide the expected portfolio return, i.e., r_0. As mentioned earlier the value of r_0 lies between r_{min} and r_{max}. To obtain the value of r_{min}, we solve the above model excluding the first constraint using the LINGO 12.0 software [105] and the computational result is summarized in Table 1.4.

Table 1.4 Summary result of portfolio selection using variance

Portfolio	Allocation				
risk	ABL	ALL	BHL	CGL	HHM
0.02631	0.0	0.0	0.0	0.0	0.56304
	HCC	KMB	MML	SIL	UNL
	0.0	0.0	0.0	0.0	0.43696

Based on the result presented in Table 1.4, we obtain the value of $r_{min} = 0.25717$. The value of r_{max} is the maximum feasible return, therefore, $r_{max} = 0.44054$. Now, by varying the value of r_0 between r_{min} and r_{max}, we solve the above model and the computational results are summarized in Table 1.5.

It can be seen that all the obtained portfolios presented in Table 1.5 are efficient portfolios. Also, it is clear from the obtained portfolios that as the return level increases, portfolio risk increases too. This relationship always hold in portfolio optimization. Fig. 1.2 shows the efficient frontier of the obtained portfolios.

It is worthy to discuss about the investor desire for portfolio return (r_0). As mentioned earlier the value of r_0 should vary between r_{min} and r_{max} and hence, if the desire is too high, i.e., $r_0 > r_{max}$, the problem P(1.1) becomes infeasible. This is so because the desire is unrealistic in the sense that it is more than what is possible from the assets under consideration. It is clear from Portfolio 10 presented in Table 1.5 that even when the investor puts the entire capital in one asset the portfolio return is 0.44054. Thus, any combination of assets can not generate a portfolio return more than 0.44054, i.e., r_{max}. On the other hand, what about the portfolio selection, if the investor desires for a portfolio return less than r_{min}? In order to handle this issue, let us assume that the investor desires for a portfolio return, $r_0 = 0.24$, i.e., less than r_{min}. We solve

Table 1.5 Summary results of portfolio selection using model P(1.1)

	Portfolio return (r_0)	Allocation										Portfolio risk
		ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL	
Portfolio 1	0.27557	0.0	0.0	0.0	0.0	0.50656	0.0	0.0	0.0	0.0	0.49344	0.02681
Portfolio 2	0.29397	0.0	0.0	0.0	0.0	0.45006	0.0	0.0	0.0	0.0	0.54994	0.02834
Portfolio 3	0.31237	0.0	0.0	0.0	0.0	0.39356	0.0	0.0	0.0	0.0	0.60644	0.03089
Portfolio 4	0.33077	0.0	0.0	0.0	0.0	0.33706	0.0	0.0	0.0	0.0	0.66294	0.03445
Portfolio 5	0.34917	0.0	0.0	0.0	0.0	0.28056	0.0	0.0	0.0	0.0	0.71944	0.03904
Portfolio 6	0.36757	0.0	0.0	0.0	0.0	0.21760	0.0	0.0	0.01160	0.0	0.77080	0.04464
Portfolio 7	0.38597	0.0	0.0	0.0	0.0	0.12107	0.0	0.0	0.08353	0.0	0.79540	0.05109
Portfolio 8	0.40437	0.0	0.0	0.0	0.0	0.02454	0.0	0.0	0.15546	0.0	0.82000	0.05832
Portfolio 9	0.42277	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.09804	0.0	0.90196	0.06673
Portfolio 10	0.44054	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	0.07689

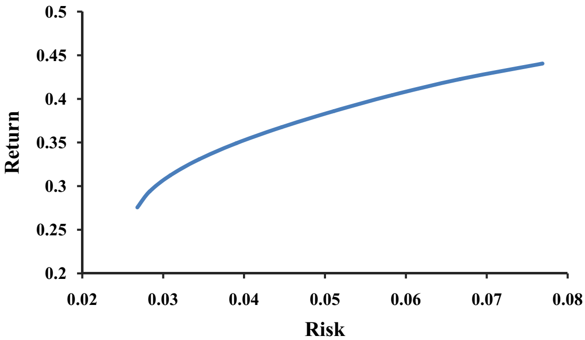

Fig. 1.2 Efficient frontier of return-risk corresponding to portfolios presented in Table 1.5

the model P(1.1) with $r_0 = 0.24$ and the computational result is presented in Table 1.6.

On comparing the portfolio presented in Table 1.6 with that given in Table 1.4 it is clear that the portfolio in Table 1.6 is not efficient. This is because the portfolio in Table 1.4 gives higher return even at lower portfolio risk.

Note that based on the information available, we can handle the mean-variance portfolio optimization problem P(1.1) in two ways. First, if the

Table 1.6 Summary result of portfolio selection using $r_0 = 0.24$

Portfolio return (r_0)	Allocation					Portfolio risk
	ABL	ALL	BHL	CGL	HHM	
0.24000	0.0	0.0	0.0	0.0	0.61578	0.02681
	HCC	KMB	MML	SIL	UNL	
	0.0	0.0	0.0	0.0	0.38422	

investor is able to specify the expected portfolio return that he/she desires, then the portfolio optimization problem P(1.1) can be solved directly to generate an efficient portfolio. Second, if the investor is not able to specify the expected portfolio return, then we can generate set of efficient portfolios using the procedure explained above and the investor may pick the one that meets his/her preferences.

Markowitz's portfolio optimization model, contrary to the theoretical reputation, has not been used extensively in its original form to construct large-scale portfolios. Its main limitations are: (i) the resultant large-scale quadratic programming problems are difficult to solve (computational complexity) and (ii) for real markets, the size of the variance-covariance matrix may be very large and hence, difficult to estimate. Several authors tried to alleviate these difficulties by using various approximation schemes. The single index model of Sharpe [110] is an early breakthrough in this direction. He pointed out that if the portfolio selection problem could be formulated as a linear programming problem, the prospects for practical applications would be greatly enhanced. Since then, many attempts have been made to linearize the portfolio optimization model. Several alternative risk measures that can be used for transformation to linear programming have been proposed in the literature. In the subsequent sections, we discuss portfolio selection models based on alternative risk measures.

1.2 Mean-Semivariance Model

Although, variance is widely accepted as a risk measure; however, it has limitations. One of the main limitations of using variance as a risk measure is that it penalizes extreme upside (gains) and downside (losses) deviations from the expected return. Thus, when probability distributions of asset returns are asymmetric, variance becomes less appropriate measure of portfolio risk [20]. This is so because the obtained portfolio may have a potential danger in terms of sacrificing higher expected return. In such cases, it is desirable to replace variance with a downside risk measure, i.e., a measure which only considers the negative deviations from a reference return level. Semivariance is one of the best known downside risk measure originally introduced by Markowitz [91] and used in mean-semivariance portfolio selection models [39, 92, 102]. Its advantage over variance is that semivariance does not consider values beyond the critical value (i.e., gains) as risk; thus, it is a more appropriate measure of risk when investors are concerned about portfolio underperformance rather than overperformance [92]. It may be noted that the implementation of mean-semivariance portfolio selection models is, however, computationally much more tedious as compared to mean-variance portfolio selection models [39, 92].

Semivariance is the expected value of the squared negative deviations of possible outcomes from the expected return. The portfolio risk measured as semivariance denoted by $s(x_1, x_2, \ldots, x_n)$ is defined as follows:

$$s(x_1, x_2, \ldots, x_n) = E\left[\left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right]^{-}\right]^2,$$

where

$$\left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right]^{-} = \begin{cases} \sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right], & \text{if } \sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right] < 0, \\ 0, & \text{if } \sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right] \geq 0. \end{cases}$$

In order to obtain portfolio selection using semivariance, it is not required to compute the variance-covariance matrix; but the joint distribution of assets is needed. This risk measure tries to minimize the dispersion of portfolio return from the expected return but only when the former is below the latter. Note that if all distribution returns are symmetric, or have the same degree of asymmetry, then semivariance and variance produces the same set of efficient portfolios [92]. Using semivariance as a risk measure, the portfolio optimization model for minimizing semivariance and constraining the expected portfolio return is formulated as follows:

$$\mathbf{P(1.3)}\quad \min\ E\left[\left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right]^{-}\right]^2$$

subject to

$$\sum_{i=1}^{n} r_i x_i = r_0\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$x_i \ge 0\,,\quad i = 1, 2, \ldots, n\,.$$

As stated earlier the expected return of the portfolio (r_0) lies between r_{min} and r_{max}. Here, r_{min} is the value r_0 of the portfolio with minimum semivariance.

Since the expected value of the random variable can be approximated by the average derived from the past data, in particular using, $r_i = E[R_i] = \sum_{t=1}^{T} r_{it}/T$, the semivariance $s(x_1, x_2, \ldots, x_n)$ is approximated as follows:

$$s(x_1, x_2, \ldots, x_n) = E\left[\left[\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right]^{-}\right]^2 = \frac{1}{T}\sum_{i=1}^{T}\left\{\left[\sum_{i=1}^{n}(r_{it} - r_i)x_i\right]^{-}\right\}^2,$$

where

$$\left[\sum_{i=1}^{n}(r_{it} - r_i)x_i\right]^{-} = \begin{cases} \sum_{i=1}^{n}(r_{it} - r_i)x_i, & \text{if } \sum_{i=1}^{n}(r_{it} - r_i)x_i < 0, \\ 0, & \text{if } \sum_{i=1}^{n}(r_{it} - r_i)x_i \ge 0, \end{cases} \qquad t = 1, 2, \ldots, T.$$

Now, the problem P(1.3) leads to the following minimization problem.

$$\mathbf{P(1.4)}\quad \min\ \frac{1}{T}\sum_{i=1}^{T}\left\{\left[\sum_{i=1}^{n}(r_{it} - r_i)x_i\right]^{-}\right\}^2$$

subject to

$$\sum_{i=1}^{n} r_i x_i = r_0\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$x_i \ge 0\,,\quad i = 1, 2, \ldots, n\,.$$

The above problem can further be transformed into the following equivalent nonlinear programming problem.

$$\textbf{P(1.5)} \quad \min \ \frac{1}{T}\sum_{i=1}^{T} p_t^2$$

subject to

$$p_t \geq -\sum_{i=1}^{n}(r_{it} - r_i)x_i, \quad t = 1,2,\ldots,T, \tag{1.7}$$

$$\sum_{i=1}^{n} r_i x_i = r_0\,, \tag{1.8}$$

$$\sum_{i=1}^{n} x_i = 1\,, \tag{1.9}$$

$$p_t \geq 0, \quad t = 1,2,\ldots,T, \tag{1.10}$$

$$x_i \geq 0\,, \quad i = 1,2,\ldots,n. \tag{1.11}$$

To understand how the problem P(1.5) can exactly solve the original mean-semivariance problem P(1.4), let us consider the following possible situations. If, for a given value t, the right-hand-side of the constraint (1.7) is negative or zero, that is, if $\sum_{i=1}^{n}(r_{it} - r_i)x_i \geq 0$, the constraint (1.7) and the constraint (1.10) leaves the variable p_t free to take any nonnegative value. Therefore, as the variable p_t^2 appears in the objective function with coefficient +1, in any optimal solution it will take value 0 since we are minimizing the sum of p_t^2. If, on the contrary, the right-hand-side of constraint (1.7) is positive, that is, if $\sum_{i=1}^{n}(r_{it} - r_i)x_i < 0$, in any optimal solution p_t will be equal to the right-hand-side value of (1.7). Thus, using the problem P(1.5) we can exactly solve the original mean-semivariance problem P(1.4).

- **Numerical illustration**

The working of the portfolio selection model P(1.5) is demonstrated with reference to the data set provided in Tables 1.1-1.2.

To find an optimal asset allocation, we now formulate the model P(1.5) as follows:

$$
\begin{aligned}
&\min\ \frac{1}{12}\cdot\left(p_1^2+p_2^2+p_3^2+p_4^2+p_5^2+p_6^2+p_7^2+p_8^2+p_9^2+p_{10}^2+p_{11}^2+p_{12}^2\right)\\
&\text{subject to}\\
&p_1-0.10299x_1-0.24383x_2-0.25313x_3-0.42223x_4+0.06747x_5-0.43689x_6\\
&\quad -0.03011x_7+0.11605x_8-0.37326x_9-0.17687x_{10}\geq 0\,,\\
&p_2+0.14533x_1+0.09083x_2+0.71634x_3+0.53118x_4+0.21513x_5+0.33237x_6\\
&\quad +0.06228x_7+0.39975x_8-0.01343x_9-0.02473x_{10}\geq 0\,,\\
&p_3+0.12210x_1+0.65083x_2-0.28463x_3-0.07463x_4+0.02191x_5+0.95560x_6\\
&\quad +0.33519x_7-0.06638x_8+0.04302x_9-0.19570x_{10}\geq 0\,,\\
&p_4+0.06101x_1-0.06516x_2-0.06413x_3+0.24977x_4+0.35047x_5+0.28078x_6\\
&\quad -0.18845x_7-0.09395x_8+0.16474x_9-0.31087x_{10}\geq 0\,,\\
&p_5-0.22661x_1-0.43530x_2-0.55818x_3-0.44366x_4-0.24261x_5-0.99053x_6\\
&\quad -0.68191x_7-0.41154x_8-0.58569x_9-0.52344x_{10}\geq 0\,,\\
&p_6+0.33134x_1+0.14750x_2+0.15254x_3+0.24177x_4+0.44580x_5+0.69345x_6\\
&\quad +0.51755x_7+0.54938x_8+0.83541x_9+0.09946x_{10}\geq 0\,,\\
&p_7-0.20016x_1+0.40018x_2+0.77537x_3+0.02053x_4-0.00648x_5+0.04850x_6\\
&\quad -0.00998x_7+0.13169x_8+0.10334x_9+0.49204x_{10}\geq 0\,,\\
&p_8+0.72984x_1+0.17083x_2+0.23634x_3+0.35827x_4-0.11487x_5+0.33592x_6\\
&\quad -0.04514x_7+0.03072x_8+0.46237x_9+0.17817x_{10}\geq 0\,,\\
&p_9-0.14285x_1-0.42735x_2-0.16836x_3-0.25978x_4+0.02835x_5-0.24703x_6\\
&\quad -0.26864x_7-0.06178x_8-0.23538x_9-0.21304x_{10}\geq 0\,,\\
&p_{10}+0.28468x_1+0.22018x_2+0.58279x_3+0.31730x_4-0.10519x_5+0.21947x_6\\
&\quad +0.45132x_7-0.04089x_8+0.49044x_9+0.24914x_{10}\geq 0\,,\\
&p_{11}+0.05201x_1+0.09984x_2-0.11613x_3-0.16323x_4-0.27253x_5-0.31722x_6\\
&\quad +0.15722x_7-0.22828x_8-0.17393x_9+0.21380x_{10}\geq 0\,,\\
&p_{12}-1.05370x_1-0.60853x_2-1.01882x_3-0.35528x_4-0.38745x_5-0.87440x_6\\
&\quad -0.29933x_7-0.32477x_8-0.71763x_9+0.21204x_{10}\geq 0\,,\\
&0.17499x_1+0.09950x_2+0.33979x_3+0.23657x_4+0.11487x_5\\
&\quad +0.27989x_6+0.21578x_7+0.25928x_8+0.26859x_9+0.44054x_{10}=r_0\,,\\
&x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8+x_9+x_{10}=1\,,\\
&p_t\geq 0\,,\quad t=1,2,\ldots,12\,,\\
&x_i\geq 0\,,\quad i=1,2,\ldots,10\,.
\end{aligned}
$$

To solve the above model, we need to decide the expected portfolio return, i.e., r_0. The value of r_{min} is obtained by solving the above model excluding the return constraint using the LINGO 12.0 and the corresponding computational result is summarized in Table 1.7.

Table 1.7 Summary result of portfolio selection using semivariance

Portfolio risk	Allocation				
	ABL	ALL	BHL	CGL	HHM
0.13628	0.0	0.0	0.0	0.0	0.61291
	HCC	KMB	MML	SIL	UNL
	0.0	0.0	0.0	0.0	0.38709

Based on the result presented in Table 1.7, we take the value of r_{min} = 0.24093. The value of r_{max} is the maximum feasible return, therefore, r_{max} = 0.44054. Now, varying the value of r_0 between r_{min} and r_{max}, we solve the above model and the corresponding computational results are summarized in Table 1.8.

Table 1.8 Summary results of portfolio selection using model P(1.5)

	Portfolio return (r_0)	Allocation					Portfolio risk
		ABL	ALL	BHL	CGL	HHM	
Portfolio 1	0.26083	0.0	0.0	0.0	0.0	0.55182	0.01389
Portfolio 2	0.28073	0.0	0.0	0.0	0.0	0.49071	0.01468
Portfolio 3	0.30063	0.0	0.0	0.0	0.0	0.42961	0.01605
Portfolio 4	0.32053	0.0	0.0	0.0	0.0	0.36850	0.01810
Portfolio 5	0.34043	0.0	0.0	0.0	0.0	0.30740	0.02074
Portfolio 6	0.36033	0.0	0.0	0.0	0.0	0.24629	0.02381
Portfolio 7	0.38023	0.0	0.0	0.0	0.0	0.18519	0.02730
Portfolio 8	0.40013	0.0	0.0	0.0	0.0	0.12408	0.03121
Portfolio 9	0.42003	0.0	0.0	0.0	0.0	0.06298	0.03556
Portfolio 10	0.44054	0.0	0.0	0.0	0.0	0.0	0.04052
		HCC	KMB	MML	SIL	UNL	
		0.0	0.0	0.0	0.0	0.44818	
		0.0	0.0	0.0	0.0	0.50929	
		0.0	0.0	0.0	0.0	0.57039	
		0.0	0.0	0.0	0.0	0.63150	
		0.0	0.0	0.0	0.0	0.69260	
		0.0	0.0	0.0	0.0	0.75371	
		0.0	0.0	0.0	0.0	0.81481	
		0.0	0.0	0.0	0.0	0.87592	
		0.0	0.0	0.0	0.0	0.93702	
		0.0	0.0	0.0	0.0	1.0	

It may be noted that all the obtained portfolios presented in Table 1.8 are efficient portfolios. Fig. 1.3 shows the efficient frontier obtained in respect of the portfolios presented in Table 1.8.

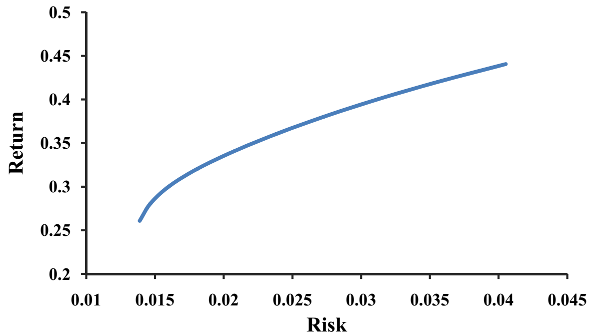

Fig. 1.3 Efficient frontier of return-risk corresponding to portfolios presented in Table 1.8

1.3 Mean-Absolute Deviation Model

To improve Markowitz's mean-variance model both computationally and theoretically, Konno and Yamazaki [72] proposed a linear programming portfolio selection model using absolute deviation as an alternative measure to quantify risk. They called the risk (absolute deviation) function as L_1-risk function because it is based on L_1 metric on R^n. It is important to point out that in metric terminology, risk (variance) function can be termed as a L_2-risk function since it is based on the notion of L_2 metric. The mathematical model proposed by Konno and Yamazaki [72] can treat the difficulties associated with the Markowitz's mean-variance model while maintaining its advantages over equilibrium models such as mean-semivariance. Much attention has been focused on this risk function because the portfolio optimization problem with L_1 risk function can be converted into a scalar parametric linear programming problem. Hence, the implementation of the portfolio optimization with this model can be easily obtained even when large number of assets are considered. Simplicity and computational ease are perceived as the most important advantages of the mean-absolute deviation model. In particular, the mean-absolute deviation model has been applied to problems with asymmetric distributions of the rate of return [126].

The absolute deviation of a random variable is the expected absolute value of the difference between the random variable and its mean. The portfolio risk measured as absolute deviation denoted by $m(x_1, x_2, \ldots, x_n)$ is expressed as follows:

$$m(x_1, x_2, \ldots, x_n) = E\left[\left|\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right|\right].$$

The following theorem proved in Konno and Yamazaki [72] presents the relationship between portfolio risk using variance and absolute deviation as risk measures.

Theorem 1.1. *Let* $(R_1, R_2, \ldots, R_n)$ *be multivariate normally distributed. Then for a given portfolio* $x = (x_1, x_2, \ldots, x_n)$

$$m(x) = \sqrt{\frac{2}{\pi}}\sigma(x)\,,$$

where

$$\sigma(x) = \sqrt{E\left[\left\{\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right\}^2\right]}\,.$$

Proof. Let $(r_1, r_2, \ldots, r_n)$ be the mean of $(R_1, R_2, \ldots, R_n)$. Also, let $(\sigma_{ij}) \in R^{n\times n}$ be the variance-covariance matrix of $(R_1, R_2, \ldots, R_n)$. Then under the given hypothesis, $\sum_{i=1}^{n} R_i x_i$ is normally distributed with mean $\sum_{i=1}^{n} r_i x_i$ and standard deviation

$$\sigma(x) = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{ij} x_i x_j}\,.$$

Therefore,

$$\begin{aligned} m(x) &= \frac{1}{\sqrt{2\pi}\sigma(x)} \int_{-\infty}^{\infty} |r|\; exp\left(-\frac{r^2}{2\sigma^2(x)}\right) dr\,, \\ &= \frac{2}{\sqrt{2\pi}\sigma(x)} \int_{0}^{\infty} r\; exp\left(-\frac{r^2}{2\sigma^2(x)}\right) dr\,, \end{aligned}$$

which on substitution $(r^2 / 2\sigma^2(x)) = s$ gives

$$m(x) = \sqrt{\frac{2}{\pi}}\frac{\sigma^2(x)}{\sigma(x)} \int_{0}^{\infty} e^{(-s)} ds = \sqrt{\frac{2}{\pi}}\sigma(x)\,.$$

□

Based on Theorem 1.1, it is clear that the mean-absolute deviation approach is equivalent to the mean-variance approach if the returns are multivariate normally distributed, i.e., both L_1 and L_2-risk models are equivalent. Even in the case when normality assumption does not hold, through certain case studies, it has been shown that minimizing the L_1-risk produces portfolios which are comparable to Markowitz's mean-variance model minimizing L_2-risk. Under these assumptions, the minimization of the sum of absolute deviations of portfolio returns about the mean is equivalent to the minimization of the variance. Thus, we have the following alternative risk minimization problem for portfolio selection.

$$\begin{aligned}
\mathbf{P(1.6)}\quad & \min\ E\left[\left|\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right|\right] \\
& \text{subject to} \\
& \sum_{i=1}^{n} r_i x_i = r_0\,, \\
& \sum_{i=1}^{n} x_i = 1\,, \\
& x_i \geq 0\,, \quad i = 1,2,\ldots,n\,.
\end{aligned}$$

Using the fact that the expected value of the random variable can be approximated by the average derived from the past data, particularly, using $r_i = E[R_i] = \sum_{t=1}^{T} r_{it}/T$, the portfolio risk $m(x_1, x_2, \ldots, x_n)$ is approximated as follows:

$$m(x_1, x_2, \ldots, x_n) = E\left[\left|\sum_{i=1}^{n} R_i x_i - E\left[\sum_{i=1}^{n} R_i x_i\right]\right|\right] = \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right|\,.$$

Now, the problem P(1.6) leads to the following minimization problem.

$$\begin{aligned}
\mathbf{P(1.7)}\quad & \min\ \frac{1}{T}\sum_{t=1}^{T}\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| \\
& \text{subject to} \\
& \sum_{i=1}^{n} r_i x_i = r_0\,, \\
& \sum_{i=1}^{n} x_i = 1\,, \\
& x_i \geq 0\,, \quad i = 1,2,\ldots,n\,.
\end{aligned}$$

The problem P(1.7) is nonlinear and nonsmooth due to the presence of absolute-valued function. In order to eliminate the absolute-valued function in problem P(1.7), we transform the problem into the following form.

$$\textbf{P(1.8)} \quad \min \ \frac{1}{T}\sum_{t=1}^{T} p_t$$

subject to

$$p_t \geq -\sum_{i=1}^{n}(r_{it} - r_i)x_i\,, \quad t = 1,2,\ldots,T, \tag{1.12}$$

$$p_t \geq \sum_{i=1}^{n}(r_{it} - r_i)x_i\,, \quad t = 1,2,\ldots,T, \tag{1.13}$$

$$\sum_{i=1}^{n} r_i x_i = r_0\,, \tag{1.14}$$

$$\sum_{i=1}^{n} x_i = 1\,, \tag{1.15}$$

$$p_t \geq 0, \quad t = 1,2,\ldots,T, \tag{1.16}$$

$$x_i \geq 0\,, \quad i = 1,2,\ldots,n\,. \tag{1.17}$$

where $p_t = \left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| = \max\left(\sum_{i=1}^{n}(r_{it} - r_i)x_i,\ -\sum_{i=1}^{n}(r_{it} - r_i)x_i\right)$. In comparison to problem P(1.7), problem P(1.8) is a linear programming problem that can be solved efficiently even when the large number of assets are considered, i.e., n is large.

To understand the role of p_t more clearly, we present the following discussion. If, for a given value t, the right-hand-side of the constraint (1.12) is negative or zero, that is, if $\sum_{i=1}^{n}(r_{it} - r_i)x_i \geq 0$, the constraint (1.12) and the constraints (1.13) and (1.16) leave the variable p_t free to take any nonnegative value. Therefore, as the variable p_t appear in the objective function with coefficient $+1$, in any optimal solution it will take value 0 since we are minimizing the sum of p_t. If, on the contrary, the right-hand-side of constraint (1.12) is positive, that is, if $\sum_{i=1}^{n}(r_{it} - r_i)x_i < 0$, in any optimal solution p_t will be equal the right-hand-side value of (1.12). Thus, using the problem P(1.8) we can exactly solve the original mean-absolute deviation problem P(1.7).

• Numerical illustration

The working of the portfolio selection model P(1.8) is demonstrated with reference to the data set provided in Tables 1.1-1.2. In order to find an optimal asset allocation, we formulate the model P(1.8) as follows:

$$\min \ \frac{1}{12} \cdot \left(p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 + p_9 + p_{10} + p_{11} + p_{12}\right)$$

subject to

$$p_1 - 0.10299x_1 - 0.24383x_2 - 0.25313x_3 - 0.42223x_4 + 0.06747x_5 - 0.43689x_6$$
$$-0.03011x_7 + 0.11605x_8 - 0.37326x_9 - 0.17687x_{10} \geq 0\,,$$
$$p_2 + 0.14533x_1 + 0.09083x_2 + 0.71634x_3 + 0.53118x_4 + 0.21513x_5 + 0.33237x_6$$
$$+0.06228x_7 + 0.39975x_8 - 0.01343x_9 - 0.02473x_{10} \geq 0\,,$$
$$p_3 + 0.12210x_1 + 0.65083x_2 - 0.28463x_3 - 0.07463x_4 + 0.02191x_5 + 0.95560x_6$$
$$+0.33519x_7 - 0.06638x_8 + 0.04302x_9 - 0.19570x_{10} \geq 0\,,$$
$$p_4 + 0.06101x_1 - 0.06516x_2 - 0.06413x_3 + 0.24977x_4 + 0.35047x_5 + 0.28078x_6$$
$$-0.18845x_7 - 0.09395x_8 + 0.16474x_9 - 0.31087x_{10} \geq 0\,,$$
$$p_5 - 0.22661x_1 - 0.43530x_2 - 0.55818x_3 - 0.44366x_4 - 0.24261x_5 - 0.99053x_6$$
$$-0.68191x_7 - 0.41154x_8 - 0.58569x_9 - 0.52344x_{10} \geq 0\,,$$
$$p_6 + 0.33134x_1 + 0.14750x_2 + 0.15254x_3 + 0.24177x_4 + 0.44580x_5 + 0.69345x_6$$
$$+0.51755x_7 + 0.54938x_8 + 0.83541x_9 + 0.09946x_{10} \geq 0\,,$$
$$p_7 - 0.20016x_1 + 0.40018x_2 + 0.77537x_3 + 0.02053x_4 - 0.00648x_5 + 0.04850x_6$$
$$-0.00998x_7 + 0.13169x_8 + 0.10334x_9 + 0.49204x_{10} \geq 0\,,$$
$$p_8 + 0.72984x_1 + 0.17083x_2 + 0.23634x_3 + 0.35827x_4 - 0.11487x_5 + 0.33592x_6$$
$$-0.04514x_7 + 0.03072x_8 + 0.46237x_9 + 0.17817x_{10} \geq 0\,,$$
$$p_9 - 0.14285x_1 - 0.42735x_2 - 0.16836x_3 - 0.25978x_4 + 0.02835x_5 - 0.24703x_6$$
$$-0.26864x_7 - 0.06178x_8 - 0.23538x_9 - 0.21304x_{10} \geq 0\,,$$
$$p_{10} + 0.28468x_1 + 0.22018x_2 + 0.58279x_3 + 0.31730x_4 - 0.10519x_5 + 0.21947x_6$$
$$+0.45132x_7 - 0.04089x_8 + 0.49044x_9 + 0.24914x_{10} \geq 0\,,$$
$$p_{11} + 0.05201x_1 + 0.09984x_2 - 0.11613x_3 - 0.16323x_4 - 0.27253x_5 - 0.31722x_6$$
$$+0.15722x_7 - 0.22828x_8 - 0.17393x_9 + 0.21380x_{10} \geq 0\,,$$
$$p_{12} - 1.05370x_1 - 0.60853x_2 - 1.01882x_3 - 0.35528x_4 - 0.38745x_5 - 0.87440x_6$$
$$-0.29933x_7 - 0.32477x_8 - 0.71763x_9 + 0.21204x_{10} \geq 0\,,$$
$$p_1 + 0.10299x_1 + 0.24383x_2 + 0.25313x_3 + 0.42223x_4 - 0.06747x_5 + 0.43689x_6$$
$$+0.03011x_7 - 0.11605x_8 + 0.37326x_9 + 0.17687x_{10} \geq 0\,,$$
$$p_2 - 0.14533x_1 - 0.09083x_2 - 0.71634x_3 - 0.53118x_4 - 0.21513x_5 - 0.33237x_6$$
$$-0.06228x_7 - 0.39975x_8 + 0.01343x_9 + 0.02473x_{10} \geq 0\,,$$
$$p_3 - 0.12210x_1 - 0.65083x_2 + 0.28463x_3 + 0.07463x_4 - 0.02191x_5 - 0.95560x_6$$
$$-0.33519x_7 + 0.06638x_8 - 0.04302x_9 + 0.19570x_{10} \geq 0\,,$$
$$p_4 - 0.06101x_1 + 0.06516x_2 + 0.06413x_3 - 0.24977x_4 - 0.35047x_5 - 0.28078x_6$$
$$+0.18845x_7 + 0.09395x_8 - 0.16474x_9 + 0.31087x_{10} \geq 0\,,$$
$$p_5 + 0.22661x_1 + 0.43530x_2 + 0.55818x_3 + 0.44366x_4 + 0.24261x_5 + 0.99053x_6$$
$$+0.68191x_7 + 0.41154x_8 + 0.58569x_9 + 0.52344x_{10} \geq 0\,,$$

$$
\begin{aligned}
&p_6 - 0.33134x_1 - 0.14750x_2 - 0.15254x_3 - 0.24177x_4 - 0.44580x_5 - 0.69345x_6 \\
&\quad -0.51755x_7 - 0.54938x_8 - 0.83541x_9 - 0.09946x_{10} \geq 0\,, \\
&p_7 + 0.20016x_1 - 0.40018x_2 - 0.77537x_3 - 0.02053x_4 + 0.00648x_5 - 0.04850x_6 \\
&\quad +0.00998x_7 - 0.13169x_8 - 0.10334x_9 - 0.49204x_{10} \geq 0\,, \\
&p_8 - 0.72984x_1 - 0.17083x_2 - 0.23634x_3 - 0.35827x_4 + 0.11487x_5 - 0.33592x_6 \\
&\quad +0.04514x_7 - 0.03072x_8 - 0.46237x_9 - 0.17817x_{10} \geq 0\,, \\
&p_9 + 0.14285x_1 + 0.42735x_2 + 0.16836x_3 + 0.25978x_4 - 0.02835x_5 + 0.24703x_6 \\
&\quad +0.26864x_7 + 0.06178x_8 + 0.23538x_9 + 0.21304x_{10} \geq 0\,, \\
&p_{10} - 0.28468x_1 - 0.22018x_2 - 0.58279x_3 - 0.31730x_4 + 0.10519x_5 - 0.21947x_6 \\
&\quad -0.45132x_7 + 0.04089x_8 - 0.49044x_9 - 0.24914x_{10} \geq 0\,, \\
&p_{11} - 0.05201x_1 - 0.09984x_2 + 0.11613x_3 + 0.16323x_4 + 0.27253x_5 + 0.31722x_6 \\
&\quad -0.15722x_7 + 0.22828x_8 + 0.17393x_9 - 0.21380x_{10} \geq 0\,, \\
&p_{12} + 1.05370x_1 + 0.60853x_2 + 1.01882x_3 + 0.35528x_4 + 0.38745x_5 + 0.87440x_6 \\
&\quad +0.29933x_7 + 0.32477x_8 + 0.71763x_9 - 0.21204x_{10} \geq 0\,, \\
&0.17499x_1 + 0.09950x_2 + 0.33979x_3 + 0.23657x_4 + 0.11487x_5 \\
&\quad +0.27989x_6 + 0.21578x_7 + 0.25928x_8 + 0.26859x_9 + 0.44054x_{10} = r_0\,, \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,, \\
&p_t \geq 0\,, \quad t = 1, 2, \ldots, 12\,, \\
&x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,.
\end{aligned}
$$

To solve the above model, we need to decide the expected portfolio return, i.e., r_0. The value of r_{min} is obtained by solving the above model excluding the return constraint using the LINGO 12.0 and the computational result is summarized in Table 1.9.

Table 1.9 Summary result of portfolio selection using absolute deviation

Portfolio	Allocation				
risk	ABL	ALL	BHL	CGL	HHM
0.12183	0.0	0.0	0.0	0.0	0.47006
	HCC	KMB	MML	SIL	UNL
	0.0	0.0	0.0	0.0	0.52994

Based on the result presented in Table 1.9, we take the value of $r_{min} = 0.28745$. The value of r_{max} is the maximum feasible return, therefore, $r_{max} = 0.44054$. Now, varying the value of r_0 between r_{min} and r_{max}, we solve the above model and and the computational results are summarized in Table 1.10.

Table 1.10 Summary results of portfolio selection using model P(1.8)

	Portfolio return (r_0)	Allocation ABL	ALL	BHL	CGL	HHM	Portfolio risk
Portfolio 1	0.30275	0.0	0.0	0.0	0.0	0.42310	0.12755
Portfolio 2	0.31805	0.0	0.0	0.0	0.0	0.37612	0.13574
Portfolio 3	0.33335	0.0	0.0	0.0	0.0	0.31995	0.14586
Portfolio 4	0.34865	0.0	0.0	0.0	0.0	0.0.2554	0.15773
Portfolio 5	0.36395	0.0	0.0	0.0	0.0	0.19085	0.16961
Portfolio 6	0.37925	0.0	0.0	0.0	0.0	0.12630	0.18148
Portfolio 7	0.39455	0.0	0.0	0.0	0.0	0.06326	0.19344
Portfolio 8	0.40985	0.0	0.0	0.0	0.0	0.01145	0.20605
Portfolio 9	0.42515	0.0	0.0	0.0	0.0	0.0	0.22098
Portfolio 10	0.44054	0.0	0.0	0.0	0.0	0.0	0.24078
		HCC	KMB	MML	SIL	UNL	
		0.0	0.0	0.0	0.0	0.57690	
		0.0	0.0	0.0	0.0	0.62388	
		0.01862	0.0	0.0	0.0	0.66143	
		0.05424	0.0	0.0	0.0	0.69036	
		0.08986	0.0	0.0	0.0	0.71929	
		0.12548	0.0	0.0	0.0	0.74822	
		0.15803	0.0	0.0	0.0	0.77871	
		0.16782	0.0	0.0	0.0	0.82073	
		0.09580	0.0	0.0	0.0	0.90420	
		0.0	0.0	0.0	0.0	1.0	

It may be noted that all the obtained portfolios presented in Table 1.10 are efficient portfolios. Fig. 1.4 shows the efficient frontier obtained in respect of the portfolios presented in Table 1.10.

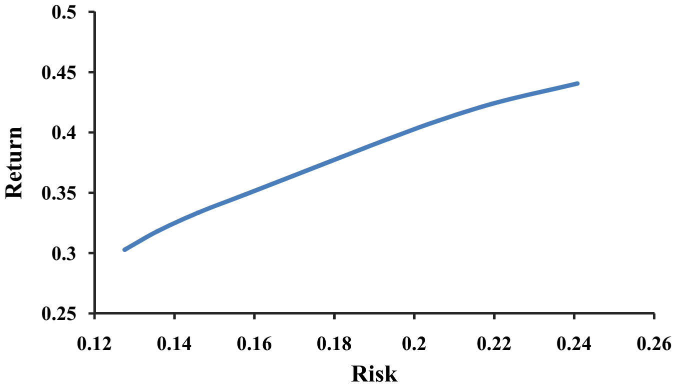

Fig. 1.4 Efficient frontier of return-risk corresponding to portfolios presented in Table 1.10

Remark 1.4. *The value of r_{min} for the mean-absolute deviation model is higher than those for mean-variance model and mean-semivariance model. More specifically, mean-absolute deviation provides higher value of r_{min} than mean-variance model and mean-variance model provides higher value of r_{min} than mean-semivariance model.*

1.4 Mean-Semiabsolute Deviation Model

Motivated by the work of Konno and Yamazaki [72], Sprenza [112] proposed semi-absolute deviation as an alternative measure to quantify risk. Speranza [112] showed that taking risk function as a linear combination of the mean semi-absolute deviations, i.e., mean deviations below and above the portfolio return, a model equivalent to the mean-absolute deviation model [72] can be obtained, whenever the sum of the coefficients of the linear combination is positive. Then, in turn, this model is equivalent to Markowitz model, if the returns are normally distributed. Moreover, Speranza showed that, through a suitable selection of the coefficients of the combination, 1 and 0 for the deviations below and above the average, respectively, it is possible to substantially reduce the number of constraints by half in comparison with the mean-absolute deviation model.

The semi-absolute deviation of return of the portfolio below the expected return over the past period t, $t = 1, 2, \ldots, T$ can be expressed as

$$w_t(x_1, x_2, \ldots, x_n) = \left| \min\left\{0, \sum_{i=1}^{n} (r_{it} - r_i)x_i \right\} \right| = \frac{\left| \sum_{i=1}^{n} (r_{it} - r_i)x_i \right| + \sum_{i=1}^{n} (r_i - r_{it})x_i}{2}.$$

Therefore, the expected semi-absolute deviation of return of the portfolio below the expected return is given by

$$\begin{aligned} w(x_1, x_2, \ldots, x_n) &= \frac{1}{T} \sum_{t=1}^{T} w_t(x_1, x_2, \ldots, x_n) \\ &= \sum_{t=1}^{T} \frac{\left| \sum_{i=1}^{n} (r_{it} - r_i)x_i \right| + \sum_{i=1}^{n} (r_i - r_{it})x_i}{2T}. \end{aligned}$$

Using semi-absolute deviation as a risk measure, the portfolio optimization model for minimizing semi-absolute deviation and constraining the expected portfolio return is formulated as follows:

$$\mathbf{P(1.9)}\quad \min \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right| + \sum_{i=1}^{n}(r_i-r_{it})x_i}{2T}$$

subject to

$$\sum_{i=1}^{n} r_i x_i = r_0\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$x_i \geq 0\,,\quad i=1,2,\ldots,n\,.$$

To eliminate the absolute-valued function in P(1.9), we transform the problem into the following form.

$$\mathbf{P(1.10)}\quad \min\ \frac{1}{T}\sum_{t=1}^{T} p_t$$

subject to

$$p_t \geq -\sum_{i=1}^{n}(r_{it}-r_i)x_i\,,\quad t=1,2,\ldots,T,$$

$$\sum_{i=1}^{n} r_i x_i = r_0\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$p_t \geq 0\,,\quad t=1,2,\ldots,T,$$

$$x_i \geq 0\,,\quad i=1,2,\ldots,n\,.$$

Remark 1.5. *The comparison of the mean-absolute deviation model P(1.8) and mean-semiabsolute deviation model P(1.10) makes it clear that mean-semiabsolute deviation model reduces the number of constraints by half in comparison with the mean-absolute deviation model. To be more explanatory, the mean-semiabsolute deviation model requires only* T *linearizing constraints whereas the mean-absolute deviation model requires* $2T$ *linearizing constraints.*

1.5 Comparison of the Models

We now compare the performance of the portfolio selection models, namely, mean-variance model, mean-semivariance model, mean-absolute deviation model. These models are solved with the data set provided in Tables 1.1-1.2 by varying the value of r_0. The computational results are summarized

Table 1.11 Summary result of portfolio selection- a comparison

	Portfolio return (r_0)	Portfolio risk		
		$\sqrt{\text{variance}}$	$\sqrt{\text{semivariance}}$	absolute deviation
Portfolio 1	0.33077	0.18561	0.13925	0.14386
Portfolio 2	0.34917	0.19759	0.14843	0.15837
Portfolio 3	0.36757	0.21128	0.15821	0.17242
Portfolio 4	0.38597	0.22603	0.16846	0.18670
Portfolio 5	0.40437	0.24150	0.17916	0.20154
Portfolio 6	0.42277	0.25832	0.19024	0.21855

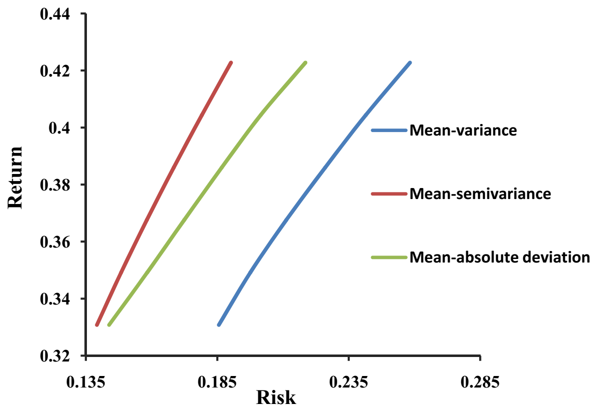

Fig. 1.5 Efficient frontier of models in respect of return-risk

in Table 1.11. Fig. 1.5 show the efficient frontiers obtained in respect of the portfolios presented in Table 1.11.

It is clear from the results presented in Table 1.11 that for the same expected return, the three models show different risk values. The risk values using mean-variance model are always higher than those obtained using mean-semivariance model and mean-absolute deviation model. Particularly, the mean-variance model provides higher risk than mean-absolute deviation model and mean-absolute deviation model provides higher risk than mean-semivariance model at any given portfolio return. These relationships exist due to the following reasons: (1) variance as a risk measure penalizes extreme upside (gains) and downside (losses) movements from the expected return. On the other hand semivariance does not consider values beyond the critical values (i.e., gains) as risk. Thus, the mean-variance model provides higher risk than mean-semivariance model; (2) the relationship between portfolio risk using variance and absolute deviation as risk measures presented in Theorem 1.1 clearly shows that the mean-variance model provides higher risk than mean-absolute deviation model; and (3) the semivariance risk measure does not consider values beyond the critical values (i.e., gains) as risk whereas in absolute deviation risk measure, we consider these values also as risk; therefore, the mean-absolute deviation model provides higher risk than mean-semivariance model.

1.6 Comments

In this chapter, we have presented the following facts:

- An overview of portfolio optimization has been presented.
- The Markowitz's mean-variance model for portfolio optimization has been discussed in detail to understand its formulation and limitations in terms of handling large scale portfolios.
- Various alternative risk measures to quantify portfolio risk in order to improve Markowitz's model both theoretically and computationally have been discussed.
- Moreover, to understand the working of the portfolio selection models, numerical illustration, based on real-world data have been provided.

Chapter 2
Portfolio Optimization with Interval Coefficients

Abstract. In this chapter, we discuss portfolio optimization models with interval coefficients, where the expected return, risk and liquidity of assets are treated as interval numbers. In addition, some realistic constraints such as number of assets held in the portfolio and the maximal and minimal fractions of the capital allocated to the various assets are considered. We present optimization models for portfolio selection in respect of three types of investment strategies, namely, conservative strategy, aggressive strategy and combination strategy.

2.1 Interval Numbers and Interval Arithmetic

The portfolio decisions are based on investor expectations, in regard to the return, risk and liquidity characteristics of the assets under contemplation. More importantly, one should know how different assets may be combined in order to have the desired return-risk-liquidity of the resultant portfolio. In real-world, the asset returns (risk, liquidity) and consequently portfolio returns (risk, liquidity) defy accurate or crisp measurement. This is because the information available about the assets/their issuers is often incomplete, the markets in which the assets are traded exhibit volatility and expert opinion might vary. Therefore, it would be more realistic to define portfolio parameters in terms of intervals rather than crisp numbers.

Let **R** be the set of all the real numbers. A closed and bounded interval in **R** is defined by

$$a = [\underline{\alpha}, \overline{\alpha}] = \{x \in \mathbf{R} : \underline{\alpha} \leq x \leq \overline{\alpha}\},$$

where $\underline{\alpha}$ is the finite lower bound and $\overline{\alpha}$ is the finite upper bound of interval a. Further, $a = [\underline{\alpha}, \overline{\alpha}]$ is called an interval number. The center and the half width (or, simply termed as 'width') of a are defined as

$$\text{Center of } a = m(a) = \frac{\underline{\alpha} + \overline{\alpha}}{2} \quad \text{and} \quad \text{Half width of } a = w(a) = \frac{\overline{\alpha} - \underline{\alpha}}{2}.$$

P. Gupta et al., *Fuzzy Portfolio Optimization*,
Studies in Fuzziness and Soft Computing 316,
DOI: 10.1007/978-3-642-54652-5_2,

The interval a can also be denoted by its center and width as

$$a = \langle m(a), w(a) \rangle = \{x \in \mathbf{R} : m(a) - w(a) \le x \le m(a) + w(a)\}.$$

Note that $m(a)$ has also been referred as *mean, center, central value or expected value* of interval a in the literature. Babad and Berliner [4] defined $m(a)$ as the *plausible value*, around which an interval a of possibilities exists. Similarly, $w(a)$ alternatively has been termed as the *spread* or *range* or *level of uncertainty* or *the extent of uncertainty* or simply, *uncertainty* of interval a. The limits $\underline{\alpha}$ and $\overline{\alpha}$ of an interval number have also been defined as the *lower* and *upper bounds* or as the *minimum* and *maximum value* or as the *infimum* and *supremum* [4, 36]. Okada and Gen [97] called them as *pessimistic* and *optimistic* values, respectively. Schjaer-Jacobsen [106, 107] used the name *worst-* and *best-case* of economic consequences for these parameters.

Remark 2.1. *An interval number can be represented by its characteristic function which takes the value* 1 *over the interval,* 0 *otherwise (see Fig. 2.1),* $\forall x \in a$ *on* $\mathbf{R}$ *as*

$$\mu_a(x) = \begin{cases} 1, & \text{if } \underline{\alpha} \le x \le \overline{\alpha}, \\ 0, & \text{otherwise}. \end{cases}$$

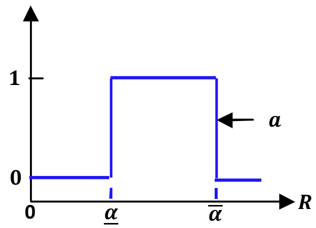

Fig. 2.1 Graphical representation of characteristic function of a

Extension of ordinary arithmetic to closed and bounded intervals of $\mathbf{R}$ is known as interval arithmetic. We, first discuss interval arithmetic, i.e., how to perform 'addition', 'subtraction', 'multiplication' and 'division' between two given closed intervals in $\mathbf{R}$. A detailed discussion on interval arithmetic is presented in Alefeld and Mayer [2] and Hansen [47].

Let $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$ be two closed and bounded intervals in $\mathbf{R}$. We present the following definitions.

Definition 2.1 (Addition (+) and subtraction (-)). Let $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$ be two interval numbers. Then, the addition of a and b, denoted by $a(+)b$, is defined as

$$a(+)b = [\underline{\alpha}, \overline{\alpha}](+)[\underline{\beta}, \overline{\beta}] = [\underline{\alpha} + \underline{\beta}, \overline{\alpha} + \overline{\beta}].$$

The addition $a(+)b$ can also be denoted by its center and width as

$$\begin{aligned} a(+)b &= \langle (m(a) + m(b)), (w(a) + w(b)) \rangle \\ &= \{x \in \mathbf{R} : (m(a) + m(b)) - (w(a) + w(b)) \leq x \leq (m(a) + m(b)) + (w(a) + w(b))\}. \end{aligned}$$

Similarly, the subtraction of a and b, denoted by $a(-)b$ is defined as

$$a(-)b = [\underline{\alpha}, \overline{\alpha}](-)[\underline{\beta}, \overline{\beta}] = [\underline{\alpha} - \overline{\beta}, \overline{\alpha} - \underline{\beta}].$$

The subtraction $a(-)b$ can also be denoted by its center and width as

$$\begin{aligned} a(-)b &= \langle (m(a) - m(b)), (w(a) + w(b)) \rangle \\ &= \{x \in \mathbf{R} : (m(a) - m(b)) - (w(a) + w(b)) \leq x \leq (m(a) - m(b)) + (w(a) + w(b))\}. \end{aligned}$$

Example 2.1. *Let $a = [4, 6]$ and $b = [3, 8]$ be two interval numbers. Then the addition and subtraction of these two interval numbers is obtained as*

$$a(+)b = [4, 6](+)[3, 8] = [4 + 3, 6 + 8] = [7, 14].$$
$$a(-)b = [4, 6](-)[3, 8] = [4 - 8, 6 - 3] = [-4, 3].$$

Fig. 2.2 depicts the graphical representation of addition and subtraction of the above defined interval numbers.

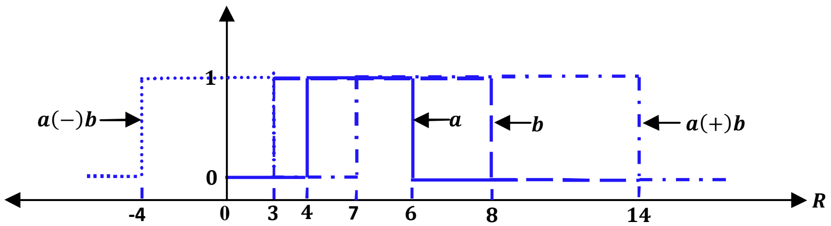

Fig. 2.2 Graphical representation of $a(+)b$ and $a(-)b$

Definition 2.2 (Image of an interval). Let $a = [\underline{\alpha}, \overline{\alpha}]$ be an interval number. Then the image of a, denoted by $\overline{a}$ is defined as

$$\overline{a} = \overline{[\underline{\alpha}, \overline{\alpha}]} = [-\overline{\alpha}, -\underline{\alpha}].$$

The image $\overline{a}$ can also be denoted by its center and width as

$$\begin{aligned} \overline{a} &= \langle -m(a), w(a) \rangle \\ &= \{x \in \mathbf{R} : -m(a) - w(a) \leq x \leq -m(a) + w(a)\}. \end{aligned}$$

Example 2.2. *Let* $a = [4, 6]$ *be an interval number. Then its image is obtained as*

$$\overline{a} = [-6, -4].$$

Fig. 2.3 depicts the graphical representation of image of the above defined interval number.

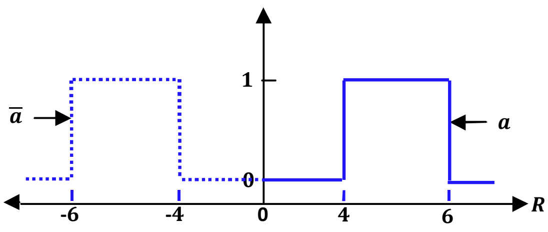

Fig. 2.3 Graphical representation of $\overline{a}$

Definition 2.3 (Multiplication (·)). Let $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$ be two interval numbers. Then their product, denoted by $a(\cdot)b$, is defined as

$$\begin{aligned} a(\cdot)b &= [\underline{\alpha}, \overline{\alpha}](\cdot)[\underline{\beta}, \overline{\beta}] \\ &= [\min(\underline{\alpha}\underline{\beta}, \underline{\alpha}\overline{\beta}, \overline{\alpha}\underline{\beta}, \overline{\alpha}\overline{\beta}), \max(\underline{\alpha}\underline{\beta}, \underline{\alpha}\overline{\beta}, \overline{\alpha}\underline{\beta}, \overline{\alpha}\overline{\beta})]. \end{aligned}$$

The product $a(\cdot)b$ can also be denoted by its center and width as

$$\begin{aligned} a(\cdot)b = \{x \in \mathbf{R} : &\min((m(a) - w(a)) \cdot (m(b) - w(b)), (m(a) - w(a)) \cdot (m(b) + w(b)), \\ &(m(a) + w(a)) \cdot (m(b) - w(b)), (m(a) + w(a)) \cdot (m(b) + w(b))) \\ &\le x \le \max((m(a) - w(a)) \cdot (m(b) - w(b)), (m(a) - w(a)) \cdot (m(b) + w(b)), \\ &(m(a) + w(a)) \cdot (m(b) - w(b)), (m(a) + w(a)) \cdot (m(b) + w(b)))\}. \end{aligned}$$

In case the intervals are in $\mathbf{R}_+$, the non-negative real line, the product $a(\cdot)b$ becomes

$$a(\cdot)b = [\underline{\alpha}\,\underline{\beta}, \overline{\alpha}\,\overline{\beta}]$$

which can also be denoted by its center and width as

$$a(\cdot)b = \{x \in \mathbf{R} : (m(a) - w(a)) \cdot (m(b) - w(b)) \le x \le (m(a) + w(a)) \cdot (m(b)+w(b))\}.$$

Example 2.3. *Let* $a = [4, 6]$ *and* $b = [3, 8]$ *be two interval numbers. Then the multiplication of these two interval numbers is obtained as*

$$\begin{aligned} a(\cdot)b &= [4,6](\cdot)[3,8] \\ &= [\min(12,32,18,48), \max(12,32,18,48)] \\ &= [12,48]. \end{aligned}$$

Fig. 2.4 depicts the graphical representation of multiplication of the above defined interval numbers.

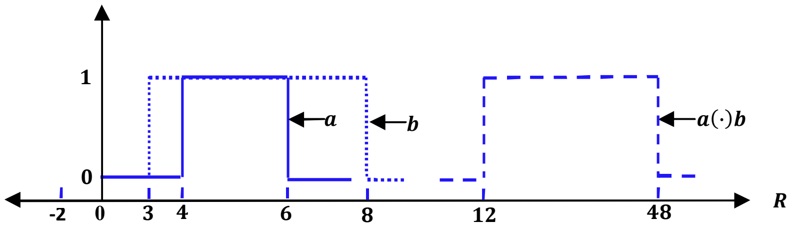

Fig. 2.4 Graphical representation of $a(\cdot)b$

Definition 2.4 (Scalar multiplication and inverse). Let $a = [\underline{\alpha}, \overline{\alpha}]$ be a closed and bounded interval in $\mathbf{R}_+$ and $k \in \mathbf{R}_+$. Considering the scalar k as the closed interval $[k,k]$, the scalar multiplication $k \cdot a$ is defined as

$$k \cdot a = [k,k](\cdot)[\underline{\alpha}, \overline{\alpha}] = [k\underline{\alpha}, k\overline{\alpha}].$$

The scalar multiplication $k \cdot a$ can also be denoted by its center and width as

$$\begin{aligned} k \cdot a &= \langle k \cdot m(a), k \cdot w(a) \rangle \\ &= \{x \in \mathbf{R} : k \cdot (m(a) - w(a)) \leq x \leq k \cdot (m(a) + w(a))\}. \end{aligned}$$

Further, the inverse of a, denoted by a^{-1}, is defined as

$$a^{-1} = [\underline{\alpha}, \overline{\alpha}]^{-1} = \left[\frac{1}{\overline{\alpha}}, \frac{1}{\underline{\alpha}}\right],$$

provided $0 \notin [\underline{\alpha}, \overline{\alpha}]$.
The inverse a^{-1} can also be denoted by its center and width as

$$\begin{aligned} a^{-1} &= \left\langle \frac{m(a)}{m(a)^2 - w(a)^2}, \frac{w(a)}{m(a)^2 - w(a)^2} \right\rangle \\ &= \left\{x \in \mathbf{R} : \frac{1}{(m(a) + w(a))} \leq x \leq \frac{1}{(m(a) - w(a))}\right\}. \end{aligned}$$

Example 2.4. *Let $a = [4,6]$ be an interval number and $k = 2$. Then the scalar multiplication is obtained as*

$$k \cdot a = [2,2](\cdot)[4,6] = [8,12].$$

Further, the inverse of a is obtained as

$$a^{-1} = [4,6]^{-1} = \left[\frac{1}{6}, \frac{1}{4}\right].$$

Fig. 2.5 depicts the graphical representation of scalar multiplication and inverse of the above defined interval number.

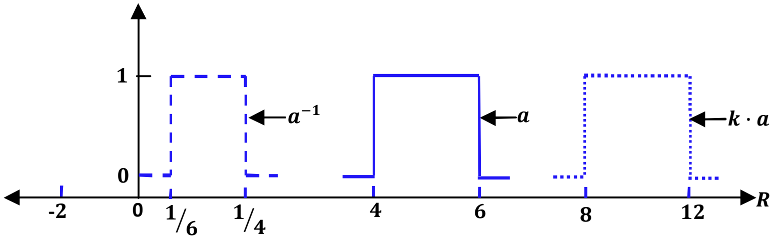

Fig. 2.5 Graphical representation of $k \cdot a$ and a^{-1}

Definition 2.5 (Division(:)). Let $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$ be two interval numbers. The division of these intervals, denoted by $a(:)b$ is defined as the multiplication of $[\underline{\alpha}, \overline{\alpha}]$ and $\left[\frac{1}{\overline{\beta}}, \frac{1}{\underline{\beta}}\right]$ provided $0 \notin [\underline{\beta}, \overline{\beta}]$. Therefore,

$$\begin{aligned} a(:)b &= [\underline{\alpha}, \overline{\alpha}](:)[\underline{\beta}, \overline{\beta}] \\ &= [\underline{\alpha}, \overline{\alpha}](\cdot)\left[\frac{1}{\overline{\beta}}, \frac{1}{\underline{\beta}}\right] \\ &= \left[\min\left(\frac{\underline{\alpha}}{\overline{\beta}}, \frac{\underline{\alpha}}{\underline{\beta}}, \frac{\overline{\alpha}}{\overline{\beta}}, \frac{\overline{\alpha}}{\underline{\beta}}\right), \max\left(\frac{\underline{\alpha}}{\overline{\beta}}, \frac{\underline{\alpha}}{\underline{\beta}}, \frac{\overline{\alpha}}{\overline{\beta}}, \frac{\overline{\alpha}}{\underline{\beta}}\right)\right]. \end{aligned}$$

The division $a(:)b$ can also be denoted by its center and width as

$$a(:)b = \left\{x \in \mathbf{R} : \min\left(\frac{m(a)-w(a)}{m(b)+w(b)}, \frac{m(a)-w(a)}{m(b)-w(b)}, \frac{m(a)+w(a)}{m(b)+w(b)}, \frac{m(a)+w(a)}{m(b)-w(b)}\right)\right.$$
$$\left. \leq x \leq \max\left(\frac{m(a)-w(a)}{m(b)+w(b)}, \frac{m(a)-w(a)}{m(b)-w(b)}, \frac{m(a)+w(a)}{m(b)+w(b)}, \frac{m(a)+w(a)}{m(b)-w(b)}\right)\right\}.$$

In case the intervals are in $\mathbf{R}_+$ and as before $0 \notin [\underline{\beta}, \overline{\beta}]$, the division $a(:)b$ becomes

$$a(:)b = \left[\frac{\underline{\alpha}}{\overline{\beta}}, \frac{\overline{\alpha}}{\underline{\beta}}\right]$$

which can also be denoted by its center and width as

$$a(:)b = \left\{x \in \mathbf{R} : \left(\frac{m(a) - w(a)}{m(b) + w(b)}\right) \leq x \leq \left(\frac{m(a) + w(a)}{m(b) - w(b)}\right)\right\} .$$

Also, one can identify $a(:)b \equiv a(\cdot)b^{-1}$ provided $0 \notin b = [\underline{\beta}, \overline{\beta}]$. Further, along the lines of scalar multiplication, the division by a scalar $k > 0$ can also be defined as

$$a(:)k = [\underline{\alpha}, \overline{\alpha}](\cdot)\left[\frac{1}{k}, \frac{1}{k}\right] = \left[\frac{\underline{\alpha}}{k}, \frac{\overline{\alpha}}{k}\right] .$$

Example 2.5. *Let $a = [4, 6]$ and $b = [3, 8]$ be two interval numbers. Then the division of these intervals is obtained as*

$$\begin{aligned} a(:)b &= [4, 6](:)[3, 8] \\ &= [4, 6](\cdot)\left[\frac{1}{8}, \frac{1}{3}\right] \\ &= \left[\min\left(\frac{4}{8}, \frac{4}{3}, \frac{6}{8}, \frac{6}{3}\right), \max\left(\frac{4}{8}, \frac{4}{3}, \frac{6}{8}, \frac{6}{3}\right)\right] \\ &= \left[\frac{4}{8}, \frac{6}{3}\right] = [0.5, 2] . \end{aligned}$$

Fig. 2.6 depicts the graphical representation of division of the above defined interval numbers.

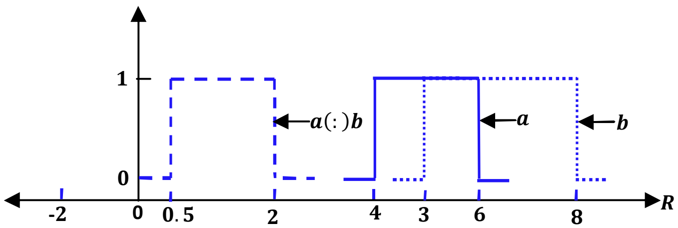

Fig. 2.6 Graphical representation of $a(:)b$

Comparison of Interval Numbers

We, now discuss the comparison between two interval numbers. There exist two types of approaches for the comparison of interval numbers, namely, set theoretic approach and probabilistic approach.

• **Set theoretic approach**

Moore [94, 95] has defined two transitive order relations between intervals $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$. The first one is an extension of '<' on the real line, defined as

$$a < b, \text{ if and only if } \overline{\alpha} < \underline{\beta}.$$

The second one is an extension of the concept of set inclusion, defined as

$$a \subseteq b, \text{ if and only if } \underline{\alpha} \geq \underline{\beta} \text{ and } \overline{\alpha} \leq \overline{\beta}.$$

A major limitation of the above mentioned order relations is that they cannot explain ranking between two overlapping intervals. The first order relation defines the simplest case of two non-overlapping intervals. The second one, describes the condition that the interval a is nested in b but it cannot order a and b neither in terms of value nor in terms of preference. Considering these facts, Ishibuchi and Tanaka [66] proposed a more prominent approach for the comparison of interval numbers. In their approach, the maximum of two interval a and b may be defied by an order relation $\leq_{LR}$ as follows:

$$a \leq_{LR} b, \text{ if and only if } \underline{\alpha} \leq \underline{\beta} \text{ and } \overline{\alpha} \leq \overline{\beta},$$
$$a <_{LR} b, \text{ if and only if } a \leq_{LR} b \text{ and } a \neq b.$$

The order relation $\leq_{LR}$ defines that if both the lower and upper limits of an interval are higher than that of another interval, then the former is higher valued interval. So, for a maximization problem in which objective function coefficients are interval profits, interval b is higher valued and hence preferred to a. Similarly, for a minimization problem in which objective function coefficients are interval costs, interval a is lower valued and hence preferred to b.

It may be noted that when one interval is nested in another, the above mentioned $\leq_{LR}$ is not applicable. For such cases, Ishibuchi and Tanaka [66] suggested another relation $\leq_{mw}$ defined as follows:

$$a \leq_{mw} b, \text{ if and only if } m(a) \leq m(b) \text{ and } w(a) \geq w(b),$$
$$a <_{mw} b, \text{ if and only if } a \leq_{mw} b \text{ and } a \neq b.$$

The second order relation $\leq_{mw}$ defines that if the central value of an interval is higher than that of another interval as well as if the latter is a wider interval, then the former interval is preferred if the problem were to choose maximum between the two. However, for a minimization problem, if a is chosen as a preferred minimum, then the condition is given as follows:

$$a \leq_{mw} b, \text{ if and only if } m(a) \leq m(b) \text{ and } w(a) \leq w(b),$$
$$a <_{mw} b, \text{ if and only if } a \leq_{mw} b \text{ and } a \neq b.$$

Example 2.6. *Let* $a = [3, 6]$ *and* $b = [4, 8]$ *be two interval returns. Then using the* $\leq_{LR}$ *order relation, we have*

$$\underline{\alpha} = 3 < \underline{\beta} = 4 \text{ and } \overline{\alpha} = 6 < \overline{\beta} = 8.$$

Therefore, b *is preferred over* a*. Further, as* $a \leq_{LR} b$ *and* $a \neq b$*, hence,* $a <_{LR} b$*.*

• Probabilistic approach

We discuss probabilistic approach given by Sengupta and Pal [108] to define an acceptability index ($\mathcal{A}$) for the comparison of intervals.

Definition 2.6 (Extended order relation). Let $\otimes$ be an extended order relation between the intervals $a = [\underline{\alpha}, \overline{\alpha}]$ and $b = [\underline{\beta}, \overline{\beta}]$ on the real line $\mathbf{R}$. Then for $m(a) \leq m(b)$, a premise $a \otimes b$ is constructed, which implies that a is *inferior* to b (or b is *superior* to a). The term 'inferior to' ('superior to') is analogous to 'less than' ('greater than').

Definition 2.7 (Acceptability index). Let I be the set of all closed and bounded intervals on the real line $\mathbf{R}$. Then, an acceptability index $\mathcal{A} : I \times I \to [0, \infty)$ is defined as

$$\mathcal{A}(a \otimes b) = \frac{m(b) - m(a)}{w(b) + w(a)},$$

where $w(b) + w(a) \neq 0$. $\mathcal{A}(a \otimes b)$ may be interpreted as *the grade of acceptability* of the '*first interval to be inferior to the second interval*'.

The grade of acceptability ($\mathcal{A}(a \otimes b)$) may be classified and interpreted further on the basis of comparative position of mean and width of interval b with respect to those of interval a as follows:

$$\mathcal{A}(a \otimes b) = \begin{cases} = 0, & \text{if } m(a) = m(b), \\ \in (0,1), & \text{if } m(a) < m(b) \text{ and } \overline{\alpha} > \underline{\beta}, \\ \geq 1, & \text{if } m(a) < m(b) \text{ and } \overline{\alpha} \leq \underline{\beta}. \end{cases}$$

The classification of the acceptability grades are interpreted as follows:

(i) If $\mathcal{A}(a \otimes b) = 0$, then the premise '*a is inferior to b*' is not accepted.
(ii) If $0 < \mathcal{A}(a \otimes b) < 1$, then the premise $(a \otimes b)$ is accepted with different grades of satisfaction ranging from zero to one (excluding zero and one).
(iii) If $\mathcal{A}(a \otimes b) \geq 1$, then the premise $(a \otimes b)$ is true.

Remark 2.2. *If $\mathcal{A}(a \otimes b) > 0$, then for a maximization problem, interval b is preferred to a and for a minimization problem, a is preferred to b in terms of value.*

Example 2.7. *Let $a = [0.10, 0.16] = \langle 0.13, 0.03 \rangle$ and $b = [0.11, 0.19] = \langle 0.15, 0.04 \rangle$ be two interval returns. Then in order to find the maximizing alternative, the investor can use the acceptability index as*

$$\mathcal{A}(a \otimes b) = \frac{m(b) - m(a)}{w(b) + w(a)} = \frac{0.15 - 0.13}{0.04 + 0.03} = 0.286.$$

Therefore, a is inferior to b, i.e., b is the maximizing alternative and the grade of satisfaction is 0.286.

2.2 Portfolio Selection Using Interval Numbers

We assume that investors allocate their wealth among n assets offering random rate of returns. We first introduce the assumptions and notation as follows.

2.2.1 Assumptions and Notation

$\widetilde{r}_i$: the expected rate of return of the i-th asset,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

r_i^{12}: the average performance of the i-th asset during a 12-month period,

r_i^{36}: the average performance of the i-th asset during a 36-month period,

r_{it}: the historical return of the i-th asset over the past period t,

$\widetilde{L}_i$: the liquidity of the i-th asset,

u_i: the maximal fraction of the capital allocated to the i-th asset,

l_i: the minimal fraction of the capital allocated to the i-th asset.

Operationally, formulating an asset portfolio requires an estimation of future returns for the various assets. Traditionally, the arithmetic mean of historical returns is considered as the expected return of an asset and thus, it is obtained as a crisp value. However, in reality, asset prices and the returns accruing therefrom are subject to a host of variables whose behavior cannot be simply extrapolated on the basis of the past. Additionally, the use of arithmetic mean of historical returns as the expected return, has two major shortcomings. Firstly, if historical data for a long period of time are considered to obtain the arithmetic mean, the influence of the earlier historical data is the same as that of the recent past data. However, recent past data of an asset are more important than the earlier historical data. Secondly, if the historical data of an asset are not adequate, due to lack of information, the estimation of the statistical parameters would not be accurate. For these reasons and to account for the uncertainty associated with estimation, the expected return of an asset is better considered as an interval number in place of the arithmetic mean of historical data. Some of the relevant references for portfolio selection using interval numbers are [8, 33, 37, 53, 54, 77, 89, 100].

Here, we present multiobjective portfolio selection models with interval coefficients. In order to determine the interval range of the expected return of an asset, we may use company's financial reports, asset's historical data and experts' judgements. To determine the range of change in expected return of an asset, we consider here the following three factors [32].

(i) **Arithmetic mean:** Although, arithmetic mean of returns of an asset should not be expressed as expected return directly, they are a good approximation. Denote the arithmetic mean return of the i-th asset as a_i, which may be calculated using historical data.

(ii) **Historical return tendency:** If recent returns of an asset have been increasing, the expected return of the asset is greater than the arithmetic mean based on historical data. If recent returns of an asset have been declining, the expected return of the asset is smaller than the arithmetic mean based on historical data. Denote the historical return tendency factor as h_i. We can use the arithmetic mean of recent returns of the i-th asset as h_i.

(iii) **Forecasted returns:** The third factor influencing the expected return of an asset is its estimated future return. Based on the publicly available information about a company, if we believe that the returns of the asset will increase then the expected return of the asset may be larger than a_i. On the other hand, if we think that returns of the asset will decrease in future, the expected return of the asset will be smaller than the arithmetic mean a_i. Denote the forecast return factor as f_i. Computation of f_i requires some forecasts based on the financial reports and experts' judgements and experiences.

Based on the above three factors, we can derive lower and upper limits of the expected return of the asset. We use the minimum of the three factors a_i, h_i and f_i as the lower limit of the expected return, while we use the maximum values of the three factors a_i, h_i and f_i as the upper limit of the expected return of the i-th asset. Therefore, the expected return of the i-th asset may be represented as the following interval number.

$$\widetilde{r}_i = \left[\underline{r}_i, \overline{r}_i\right] = [\min\{a_i, h_i, f_i\}, \max\{a_i, h_i, f_i\}] \ .$$

We consider the following objective functions and constraints in the multiobjective portfolio selection problem.

2.2.2 Objective Functions

Return

The return of the portfolio is expressed as

$$\widetilde{f}_1(x) = \sum_{i=1}^{n} \widetilde{r}_i x_i = \sum_{i=1}^{n} \left[\underline{r}_i, \overline{r}_i\right] x_i = \left[\sum_{i=1}^{n} \underline{r}_i x_i, \sum_{i=1}^{n} \overline{r}_i x_i\right] .$$

Risk
For a given expected return, the investor penalizes negative semi-absolute deviation which is defined as portfolio risk. The semi-absolute deviation of return of the portfolio below the expected return over the past period t, $t = 1, 2, \dots, T$, can be expressed as

$$w_t(x) = \left| \min\left\{0, \sum_{i=1}^{n} (r_{it} - r_i)x_i \right\} \right| = \frac{\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2}.$$

Because the expected return of assets are considered as interval numbers, we may consider the expected semi-absolute deviation of return of the portfolio $x = (x_1, x_2, \dots, x_n)$ below the expected return as an interval number too. The expected semi-absolute deviation of return of the portfolio below the expected return thus becomes

$$\begin{aligned}\widetilde{w}_t(x) &= \left[\left| \min\left\{0, \sum_{i=1}^{n} (r_{it} - \underline{r}_i)x_i \right\} \right|, \left| \min\left\{0, \sum_{i=1}^{n} (r_{it} - \overline{r}_i)x_i \right\} \right|\right] \\ &= \left[\frac{\left|\sum_{i=1}^{n}(r_{it} - \underline{r}_i)x_i\right| + \sum_{i=1}^{n}(\underline{r}_i - r_{it})x_i}{2}, \frac{\left|\sum_{i=1}^{n}(r_{it} - \overline{r}_i)x_i\right| + \sum_{i=1}^{n}(\overline{r}_i - r_{it})x_i}{2}\right].\end{aligned}$$

Therefore, the expected semi-absolute deviation of return of the portfolio $x = (x_1, x_2, \dots, x_n)$ below the expected return becomes

$$\begin{aligned}\widetilde{f_2}(x) &= \widetilde{w}(x) = \frac{1}{T}\sum_{t=1}^{T} \widetilde{w}_t(x) \\ &= \left[\sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - \underline{r}_i)x_i\right| + \sum_{i=1}^{n}(\underline{r}_i - r_{it})x_i}{2T}, \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - \overline{r}_i)x_i\right| + \sum_{i=1}^{n}(\overline{r}_i - r_{it})x_i}{2T}\right].\end{aligned}$$

We use $\widetilde{w}(x)$ to measure portfolio risk.

Liquidity
For any asset, liquidity may be measured using the turnover rate as characterized by the ratio of the average trading volume of the stocks traded in the market and the trading volume of the tradable stock (i.e., shares held by the public) corresponding to that asset. Because of incomplete information, the turnover rates are only vague estimates, therefore, we may consider the liquidity of the portfolio as an interval number too. The liquidity of the portfolio is expressed as

$$\widetilde{f_3}(x) = \sum_{i=1}^{n} \widetilde{L_i} x_i = \sum_{i=1}^{n} \left[\underline{L_i}, \overline{L_i}\right] x_i = \left[\sum_{i=1}^{n} \underline{L_i} x_i, \sum_{i=1}^{n} \overline{L_i} x_i\right].$$

2.2.3 Constraints

The short term return of the portfolio is expressed as

$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st},$$

where $r_i^{12} = \frac{1}{12}\sum_{t=1}^{12} r_{it}$, $i = 1, 2, \ldots, n$ and r_{st} is the minimum desired level of short term return indicated by the investor.
The long return of the portfolio is expressed as

$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt},$$

where $r_i^{36} = \frac{1}{36}\sum_{t=1}^{36} r_{it}$, $i = 1, 2, \ldots, n$ and r_{lt} is the minimum desired level of long term return indicated by the investor.

From the discussion on the various factors accounting for the rate of change in expected return of the assets, it is clear that the short term return (comparable with recent return h_i) and long term return (comparable with arithmetic mean a_i) have a huge impact on the portfolio return. Hence, in addition to the return objective function, it may be good to consider also the short and long term returns separately as constraints in the model since many investors may plan their asset allocation considering a minimum desired level of short term return, long term return or both [8].
Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1.$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \leq u_i y_i, \quad i = 1, 2, \ldots, n.$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \geq l_i y_i, \quad i = 1, 2, \ldots, n.$$

The maximal and minimal fractions of the capital allocated to the assets in the portfolio depend upon a number of factors. For example, one may consider the price or value of the asset relative to the average price or value of all the assets in the chosen portfolio, the minimal lot size that can be traded in the market, the past behavior of the price or volume of the asset, the information available about the issuer of the asset and trends in the industry of which it is a part. Thus, the investor considers many fundamental and technical analysis factors that affect the company and the industry. Because investors differ in their interpretation of the available information, they may allocate the same capital budget differently. The constraints corresponding to lower bounds l_i and upper bounds u_i on the investment in individual assets ($0 \leq l_i, u_i \leq 1$, $l_i \leq u_i, i = 1, 2, \ldots, n$) are included to avoid a large number of very small investments (this is ensured by the lower bounds) and to achieve sufficient diversification of the investments (this is ensured by the upper bounds); however, the lower and upper bounds have to be chosen carefully so that the portfolio selection problem does not become infeasible.

Number of assets held in a portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h\,,$$

where h is the number of assets that the investor chooses to include in the portfolio. Of all the assets from a given set, the investor would pick up the ones that are likely to yield the desired satisfaction of his/her preferences. It is not necessary that all the assets from a given set may configure in the portfolio as well. Investors would differ with respect to the number of assets they can effectively manage in a portfolio.

No short selling of assets is expressed as

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

2.2.4 The Decision Problem

The constrained multiobjective portfolio selection problem involving interval coefficients is formulated as follows:

P(2.1) $\max \widetilde{f_1}(x) = \left[\sum_{i=1}^{n} \underline{r}_i x_i, \sum_{i=1}^{n} \overline{r}_i x_i \right]$

$$\min \widetilde{f_2}(x) = \widetilde{w}(x)$$

$$= \left[\sum_{t=1}^{T} \frac{\left| \sum_{i=1}^{n} (r_{it} - \underline{r}_i) x_i \right| + \sum_{i=1}^{n} (\underline{r}_i - r_{it}) x_i}{2T}, \sum_{t=1}^{T} \frac{\left| \sum_{i=1}^{n} (r_{it} - \overline{r}_i) x_i \right| + \sum_{i=1}^{n} (\overline{r}_i - r_{it}) x_i}{2T} \right]$$

$$\max \widetilde{f_3}(x) = \left[\sum_{i=1}^{n} \underline{L}_i x_i, \sum_{i=1}^{n} \overline{L}_i x_i\right]$$

subject to

$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st}\,,$$

$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt}\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$\sum_{i=1}^{n} y_i = h\,,$$

$$x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,,$$

$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,.$$

The absolute-valued function in problem P(2.1) can be eliminated by using the transformation discussed in Chapter 1. The equivalent problem is formulated as

P(2.2)

$$\max \widetilde{f_1}(x) = \left[\sum_{i=1}^{n} \underline{r}_i x_i, \sum_{i=1}^{n} \overline{r}_i x_i\right]$$

$$\min \widetilde{f_2}(p) = \widetilde{w}(p) = \left[\frac{1}{T}\sum_{t=1}^{T} p_t^1, \frac{1}{T}\sum_{t=1}^{T} p_t^2\right]$$

$$\max \widetilde{f_3}(x) = \left[\sum_{i=1}^{n} \underline{L}_i x_i, \sum_{i=1}^{n} \overline{L}_i x_i\right]$$

subject to

$$p_t^1 + \sum_{i=1}^{n} (r_{it} - \underline{r}_i) x_i \geq 0, \; t = 1, 2, \ldots, T\,,$$

$$p_t^2 + \sum_{i=1}^{n} (r_{it} - \overline{r}_i) x_i \geq 0, \; t = 1, 2, \ldots, T\,,$$

$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st}\,,$$

$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt}\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$
$$\sum_{i=1}^{n} y_i = h\,,$$
$$x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$
$$x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$
$$x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,,$$
$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,,$$
$$p_t^1 \ge 0\,, \quad t = 1, 2, \ldots, T\,,$$
$$p_t^2 \ge 0\,, \quad t = 1, 2, \ldots, T\,.$$

2.3 Solution Methodology

The problem P(2.2) is a multiobjective mixed integer interval linear programming problem. The weighted-sum method is used to convert the multiobjective problem into the following single objective optimization problem.

P(2.3) $\max\left(\alpha \cdot \widetilde{f_1}(x) - \beta \cdot \widetilde{f_2}(x) + \gamma \cdot \widetilde{f_3}(x)\right)$

subject to

$$p_t^1 + \sum_{i=1}^{n} (r_{it} - \underline{r}_i) x_i \ge 0,\ t = 1, 2, \ldots, T\,,$$
$$p_t^2 + \sum_{i=1}^{n} (r_{it} - \overline{r}_i) x_i \ge 0,\ t = 1, 2, \ldots, T\,,$$
$$\sum_{i=1}^{n} r_i^{12} x_i \ge r_{st}\,,$$
$$\sum_{i=1}^{n} r_i^{36} x_i \ge r_{lt}\,,$$
$$\sum_{i=1}^{n} x_i = 1\,,$$
$$\sum_{i=1}^{n} y_i = h\,,$$
$$x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$
$$x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$
$$x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,,$$
$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,,$$

$$p_t^1 \geq 0\,, \quad t = 1, 2, \ldots, T\,,$$
$$p_t^2 \geq 0\,, \quad t = 1, 2, \ldots, T\,,$$

where $0 < \alpha, \beta, \gamma < 1$. If x^* is an optimal portfolio obtained by solving the problem P(2.3) then x^* is an efficient portfolio. It is worthy to mention that the weighted-sum method [35] is one of the possible ways to solve the multiobjective optimization problem. There are other solution methods, e.g. ϵ-constraint method [17], goal programming [18] and its variants-preemptive (lexicographic goal programming) [55] and non-preemptive (weighted goal programming) [93] for the purpose.
Let us denote

$$\underline{F}(x) = \alpha \cdot \left(\sum_{i=1}^{n} \underline{r}_i x_i \right) - \beta \cdot \left(\frac{1}{T} \sum_{t=1}^{T} p_t^2 \right) + \gamma \cdot \left(\sum_{i=1}^{n} \underline{L}_i x_i \right), \tag{2.1}$$

$$\overline{F}(x) = \alpha \cdot \left(\sum_{i=1}^{n} \overline{r}_i x_i \right) - \beta \cdot \left(\frac{1}{T} \sum_{t=1}^{T} p_t^1 \right) + \gamma \cdot \left(\sum_{i=1}^{n} \overline{L}_i x_i \right). \tag{2.2}$$

Then we have,

$$\max \left(\alpha \cdot \widetilde{f_1}(x) - \beta \cdot \widetilde{f_2}(x) + \gamma \cdot \widetilde{f_3}(x) \right) = \max \left[\underline{F}(x), \overline{F}(x) \right].$$

Thus, the objective function of model P(2.3) becomes an interval $\left[\underline{F}(x), \overline{F}(x) \right]$ satisfying equations (2.1)-(2.2) and the maximization may be interpreted as an optimization problem defined on the basis of some order relations between intervals. We now discuss optimization models for portfolio selection in respect of three types of investment strategies, namely, conservative strategy, aggressive strategy and combination strategy.

• **Conservative strategy**
The investor pursuing conservative strategy is more concerned about portfolio risk in comparison to portfolio return and liquidity. In other words, he/she is more concerned about reducing portfolio risk rather than enhancing portfolio return and liquidity. Hence, the optimization model for such an investor type is formulated as follows:

P(2.4) $\max \underline{F}(x)$

subject to

$$p_t^2 + \sum_{i=1}^{n} (r_{it} - \overline{r}_i) x_i \geq 0, \; t = 1, 2, \ldots, T\,,$$

$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st}\,,$$

$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt} \,,$$
$$\sum_{i=1}^{n} x_i = 1 \,,$$
$$\sum_{i=1}^{n} y_i = h \,,$$
$$x_i \leq u_i y_i \,, \quad i = 1, 2, \ldots, n \,,$$
$$x_i \geq l_i y_i \,, \quad i = 1, 2, \ldots, n \,,$$
$$x_i \geq 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$y_i \in \{0, 1\} \,, \quad i = 1, 2, \ldots, n \,,$$
$$p_t^2 \geq 0 \,, \quad t = 1, 2, \ldots, T \,.$$

- **Aggressive strategy**

The investor pursuing aggressive strategy aspires more for portfolio return and liquidity in comparison to portfolio risk. In other words, he/she is more concerned about increasing portfolio return and liquidity rather than reducing portfolio risk. Hence, the optimization model for such an investor type is formulated as follows:

P(2.5) $\max \overline{F}(x)$

subject to

$$p_t^1 + \sum_{i=1}^{n} (r_{it} - \underline{r}_i) x_i \geq 0, \; t = 1, 2, \ldots, T \,,$$
$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st} \,,$$
$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt} \,,$$
$$\sum_{i=1}^{n} x_i = 1 \,,$$
$$\sum_{i=1}^{n} y_i = h \,,$$
$$x_i \leq u_i y_i \,, \quad i = 1, 2, \ldots, n \,,$$
$$x_i \geq l_i y_i \,, \quad i = 1, 2, \ldots, n \,,$$
$$x_i \geq 0 \,, \quad i = 1, 2, \ldots, n \,,$$
$$y_i \in \{0, 1\} \,, \quad i = 1, 2, \ldots, n \,,$$
$$p_t^1 \geq 0 \,, \quad t = 1, 2, \ldots, T \,.$$

• Combination strategy

In contrast to conservative and aggressive strategies, the combination strategy is neither too conservative nor too aggressive. The investor aspires for a balance in portfolio performance on return, risk and liquidity. In other words, he/she is equally concerned about reducing portfolio risk as well as enhancing portfolio return and liquidity. Hence, the optimization model for such an investor type is formulated as follows:

P(2.6) $\max \lambda \cdot \underline{F}(x) + (1-\lambda) \cdot \overline{F}(x)$

subject to

$$p_t^1 + \sum_{i=1}^{n} (r_{it} - \underline{r}_i) x_i \geq 0, \ t = 1, 2, \ldots, T,$$

$$p_t^2 + \sum_{i=1}^{n} (r_{it} - \overline{r}_i) x_i \geq 0, \ t = 1, 2, \ldots, T,$$

$$\sum_{i=1}^{n} r_i^{12} x_i \geq r_{st},$$

$$\sum_{i=1}^{n} r_i^{36} x_i \geq r_{lt},$$

$$\sum_{i=1}^{n} x_i = 1,$$

$$\sum_{i=1}^{n} y_i = h,$$

$$x_i \leq u_i y_i, \quad i = 1, 2, \ldots, n,$$

$$x_i \geq l_i y_i, \quad i = 1, 2, \ldots, n,$$

$$x_i \geq 0, \quad i = 1, 2, \ldots, n,$$

$$y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n,$$

$$p_t^1 \geq 0, \quad t = 1, 2, \ldots, T,$$

$$p_t^2 \geq 0, \quad t = 1, 2, \ldots, T,$$

$$0 \leq \lambda \leq 1,$$

where λ is an index of pessimism ranging on a scale from 0 to 1. An $\lambda = 0$ indicates that the investor follows the aggressive strategy while $\lambda = 1$ indicates that the investor follows conservative strategy. In general, for combination strategy we choose $\lambda = 0.5$, i.e., the investor strategy is neither too conservative nor too aggressive.

2.4 Numerical Illustration

We present the results of an empirical study done for the imaginary investors using the data set extracted from NSE, Mumbai, India in order to demonstrate the working of the portfolio selection models presented in this chapter.

Ten assets listed on NSE have been randomly selected to form a population from which we attempt to construct portfolios comprising 5 assets for numerical experiments. It may be noted that it is not advisable to have very few or very large number of assets in the portfolio so as to achieve diversification. Generally, portfolio diversification by investors lies in the narrow range of 3-10 assets. The historical data of the 10 assets from April 1, 2005 to March 31, 2008 in respect of daily asset price and daily turnover rates was collected and the average returns based on the average of the averages, that is, the average monthly returns were used to obtain historical rates of return. We also use daily turnover rates to calculate the historical turnover rates, during the 36-month period. Finally, following to the method described in Section 2.2.1, we obtain the intervals in respect of return, risk and liquidity of each asset. Table 2.1 provides the data corresponding to expected return, risk and liquidity as interval numbers. The short term return and long term return of the assets are provided in Table 2.2. Also, the returns for the entire period of the study in respect of each asset are provided in Table 2.3.

Table 2.1 Input data of assets corresponding to expected return, risk and liquidity

Company	Return	Risk	Liquidity
A B B Ltd. (ABL)	[0.17499, 0.19278]	[0.12267, 0.13233]	[0.00032, 0.00050]
Ambuja Cements Ltd. (ACL)	[0.14240, 0.14868]	[0.12047, 0.12344]	[0.00201, 0.00441]
Ashok Leyland Ltd. (ALL)	[0.15058, 0.17240]	[0.17243, 0.18333]	[0.00359, 0.00706]
Bajaj Auto Ltd. (BAL)	[0.17354, 0.24587]	[0.11865, 0.15696]	[0.00152, 0.00193]
C E S C Ltd. (CSL)	[0.15108, 0.27477]	[0.13568, 0.19649]	[0.00413, 0.00424]
G A I L (India) Ltd. (GIL)	[0.05292, 0.11579]	[0.11835, 0.14229]	[0.00266, 0.00532]
H D F C Bank Ltd. (HBL)	[0.10273, 0.12124]	[0.08701, 0.09540]	[0.00132, 0.00170]
Hindustan Unilever Ltd. (HUL)	[0.06188, 0.16679]	[0.13386, 0.19557]	[0.00168, 0.00192]
Reliance Industries Ltd. (RIL)	[0.16188, 0.29220]	[0.11311, 0.18762]	[0.00672, 0.00724]
Voltas Ltd. (VOL)	[0.30120, 0.41967]	[0.16652, 0.23555]	[0.00029, 0.00039]

Table 2.2 Input data of assets corresponding to short term return, long term return, lower bound and upper bounds on allocation in each asset

Company	Short term return	Long term return	Lower bound	Upper bound
ABL	0.17499	0.19278	0.08	0.3
ACL	0.14240	0.14868	0.08	0.3
ALL	0.17240	0.15058	0.08	0.3
BAL	0.24587	0.17354	0.08	0.3
CSL	0.15108	0.27477	0.08	0.3
GIL	0.05292	0.11579	0.08	0.3
HBL	0.10273	0.12072	0.08	0.3
HUL	0.16679	0.06188	0.08	0.3
RIL	0.29220	0.16188	0.08	0.3
VOL	0.41967	0.30120	0.08	0.3

Table 2.3 Returns of the assets for the period April 1, 2005 to March 31, 2008

	Monthly returns																	
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
ABL	0.28767	0.08774	0.76645	0.01700	0.37000	0.01367	0.78129	0.03290	0.13179	0.45323	-0.02167	-0.60968	0.17367	0.18290	0.22935	-0.02667	0.04935	0.75700
ACL	0.50533	0.33419	0.13097	0.00733	0.31548	0.38133	0.27484	-0.12484	0.24929	-0.10968	0.40033	-0.48161	0.04500	-0.05032	0.54194	0.13533	0.05484	0.25267
ALL	0.51600	0.18484	0.64516	0.53833	0.62129	0.21733	0.67935	-0.75935	0.50464	-0.09742	0.19700	-0.64258	0.34233	-0.24516	-0.31097	0.00867	-0.00806	0.32867
BAL	0.38500	0.44839	0.44323	0.27900	0.49613	0.23300	0.48516	-0.38935	-0.32750	0.04290	0.04400	-0.16677	0.05633	-0.12484	0.40903	0.33000	-0.20548	0.16467
CSL	1.78900	0.29387	0.48968	-0.18800	2.36355	1.09800	0.28452	-0.15645	-0.27286	-0.25806	0.37233	-0.32484	-0.39300	0.86839	-0.05419	0.20900	0.33871	0.45167
GIL	0.26133	0.29774	0.35290	0.46433	0.34903	0.27533	1.32129	-0.37839	-0.26893	0.19194	0.11033	-0.74452	0.28333	0.47581	-0.18419	0.18633	0.08097	0.37700
HBL	0.18333	0.12419	0.13323	0.00100	0.46774	-0.14000	0.62581	-0.17452	0.30536	0.05032	-0.00433	-0.02387	0.18633	0.05065	-0.05968	0.30500	0.10806	0.60933
HUL	0.45167	-0.11355	0.33323	-0.01333	-0.15742	0.08100	0.44968	-0.25903	-0.26429	-0.36323	-0.28567	0.02516	-0.23700	-0.28516	-0.03645	0.38533	-0.00968	0.50033
RIL	0.29600	0.32065	0.37129	0.35933	0.31097	0.01367	0.53548	-0.05258	-0.02393	-0.08548	-0.06067	-0.54129	0.01800	0.42968	-0.08000	0.29033	0.07871	-0.07033
VOL	1.11800	0.31065	0.55097	-0.21667	0.46194	0.96167	0.10839	-0.64839	0.21536	-0.05387	0.34100	-0.59452	0.08600	0.49871	0.42968	0.47467	-0.07548	0.63533

	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
	0.06645	0.01645	0.98448	-0.31000	0.13900	0.26774	0.07200	0.32032	0.29710	0.23600	-0.05161	0.50633	-0.02516	0.90484	0.03214	0.45968	0.22700	-0.87871
	0.29419	0.37613	0.00448	-0.36484	0.17600	0.29548	-0.07133	0.29000	0.09161	0.56000	-0.31161	0.51100	-0.01710	0.34806	-0.00321	0.52097	0.64167	-0.85129
	0.60581	-0.03258	-0.04517	-0.36290	0.32533	0.14161	-0.06433	0.86903	-0.05645	-0.02633	-0.08355	0.44100	0.07484	-0.11742	0.84071	0.18097	0.88000	-0.86968
	0.37903	-0.25677	-0.01379	0.15935	0.00667	0.41968	0.46100	0.15710	-0.06419	0.62033	0.04387	0.55767	-0.00677	0.24387	0.69107	0.18355	0.33000	-0.26710
	0.61290	0.05484	0.49138	0.09032	-0.15767	0.07548	0.25333	0.24129	0.22774	0.05933	-0.60032	0.57200	0.09903	0.38290	0.23250	0.78613	-0.15967	-0.28129
	0.25032	0.02871	0.17000	-0.40258	-0.12233	0.15774	0.21667	-0.01548	0.11161	0.47167	-0.34968	0.44167	-0.00355	0.38419	-0.21679	0.50968	-0.31400	-0.60097
	0.15742	0.28548	0.14414	-0.22968	-0.05000	0.05774	0.56700	0.31806	-0.27774	0.23833	-0.38290	0.42733	0.11000	0.24548	-0.10964	0.16516	0.23033	-0.29871
	-0.01645	0.35774	-0.33931	-0.27323	0.15367	0.14226	0.52433	0.05645	0.03968	0.31400	-0.35839	0.41667	0.26290	-0.02613	0.82107	0.36710	0.24800	-0.66419
	0.11484	-0.00161	0.15690	-0.04935	-0.10033	0.09097	0.62500	0.30065	0.08484	0.33667	-0.12194	0.29733	0.21839	0.73258	-0.02036	0.37613	0.86167	-0.18452
	0.39419	0.23452	0.19345	0.06935	0.21233	0.09968	-0.21333	0.25032	1.65032	0.61867	-0.24903	0.81200	0.31839	0.38161	0.74107	0.71742	-0.00200	0.01065

We now present the computational results for investors pursuing three different investment strategies.

• **Conservative strategy**

Using the input data from Tables 2.1-2.3, $r_{st} = 0.165$, $r_{lt} = 0.195$, $h = 5$ and $\alpha = \beta = \gamma = 1/3$, we formulate the problem P(2.4) as follows:

$$\begin{aligned}\max \underline{F}(x) = \frac{1}{3} \cdot (&0.17499x_1 + 0.14240x_2 + 0.15058x_3 + 0.17354x_4 + 0.15108x_5 \\ &+0.05292x_6 + 0.10273x_7 + 0.06188x_8 + 0.16188x_9 + 0.30120x_{10}) \\ &-\frac{1}{3} \cdot \Big(\frac{1}{36} \cdot (p_1^2 + p_2^2 + p_3^2 + p_4^2 + p_5^2 + p_6^2 + p_7^2 + p_8^2 + p_9^2 + p_{10}^2 + p_{11}^2 + p_{12}^2 \\ &+p_{13}^2 + p_{14}^2 + p_{15}^2 + p_{16}^2 + p_{17}^2 + p_{18}^2 + p_{19}^2 + p_{20}^2 + p_{21}^2 + p_{22}^2 + p_{23}^2 + p_{24}^2 \\ &+p_{25}^2 + p_{26}^2 + p_{27}^2 + p_{28}^2 + p_{29}^2 + p_{30}^2 + p_{31}^2 + p_{32}^2 + p_{33}^2 + p_{34}^2 + p_{35}^2 + p_{36}^2)\Big) \\ &+\frac{1}{3} \cdot (0.00032x_1 + 0.00201x_2 + 0.00359x_3 + 0.00152x_4 + 0.00413x_5 \\ &+0.00266x_6 + 0.00132x_7 + 0.00168x_8 + 0.00672x_9 + 0.00029x_{10})\end{aligned}$$

subject to

$$\begin{aligned}
&p_1^2 + 0.09489x_1 + 0.35665x_2 + 0.34360x_3 + 0.13913x_4 + 1.51423x_5 + 0.14554x_6 \\
&\quad +0.06209x_7 + 0.28488x_8 + 0.00380x_9 + 0.69833x_{10} \geq 0\,, \\
&p_2^2 - 0.10504x_1 + 0.18551x_2 + 0.01244x_3 + 0.20252x_4 + 0.01911x_5 + 0.18195x_6 \\
&\quad +0.00295x_7 - 0.28034x_8 + 0.02844x_9 - 0.10903x_{10} \geq 0\,, \\
&p_3^2 + 0.57367x_1 - 0.01772x_2 + 0.47276x_3 + 0.19736x_4 + 0.21491x_5 + 0.23711x_6 \\
&\quad +0.01199x_7 + 0.16643x_8 + 0.07909x_9 + 0.13129x_{10} \geq 0\,, \\
&p_4^2 - 0.17578x_1 - 0.14135x_2 + 0.36593x_3 + 0.03313x_4 - 0.46277x_5 + 0.34854x_6 \\
&\quad -0.12024x_7 - 0.18012x_8 + 0.06713x_9 - 0.63634x_{10} \geq 0\,, \\
&p_5^2 + 0.17722x_1 + 0.16680x_2 + 0.44889x_3 + 0.25026x_4 + 2.08878x_5 + 0.23324x_6 \\
&\quad +0.34650x_7 - 0.32421x_8 + 0.01876x_9 + 0.04226x_{10} \geq 0\,, \\
&p_6^2 - 0.17911x_1 + 0.23265x_2 + 0.04493x_3 - 0.01287x_4 + 0.82323x_5 + 0.15954x_6 \\
&\quad -0.26124x_7 - 0.08579x_8 - 0.27854x_9 + 0.54199x_{10} \geq 0\,, \\
&p_7^2 + 0.58851x_1 + 0.12615x_2 + 0.50696x_3 + 0.23930x_4 + 0.00975x_5 + 1.20550x_6 \\
&\quad +0.50457x_7 + 0.28289x_8 + 0.24328x_9 - 0.31129x_{10} \geq 0\,, \\
&p_8^2 - 0.15988x_1 - 0.27352x_2 - 0.93175x_3 - 0.63522x_4 - 0.43122x_5 - 0.49418x_6 \\
&\quad -0.29576x_7 - 0.42582x_8 - 0.34478x_9 - 1.06806x_{10} \geq 0\,, \\
&p_9^2 - 0.06099x_1 + 0.10060x_2 + 0.33224x_3 - 0.57337x_4 - 0.54762x_5 - 0.38472x_6 \\
&\quad +0.18412x_7 - 0.43108x_8 - 0.31613x_9 - 0.20432x_{10} \geq 0\,, \\
&p_{10}^2 + 0.26045x_1 - 0.25836x_2 - 0.26982x_3 - 0.20296x_4 - 0.53283x_5 + 0.07614x_6 \\
&\quad -0.07092x_7 - 0.53002x_8 - 0.37769x_9 - 0.47354x_{10} \geq 0\,, \\
&p_{11}^2 - 0.21445x_1 + 0.25165x_2 + 0.02460x_3 - 0.20187x_4 + 0.09757x_5 - 0.00546x_6 \\
&\quad -0.12557x_7 - 0.45246x_8 - 0.35287x_9 - 0.07867x_{10} \geq 0\,,
\end{aligned}$$

$$p_{12}^2 - 0.41264x_1 - 0.80246x_2 - 0.14511x_3 - 0.63030x_4 - 0.59960x_5 - 0.14163x_6$$
$$-0.81498x_7 - 0.83349x_8 - 1.01419x_9 - 0.86031x_{10} \geq 0\,,$$
$$p_{13}^2 - 0.01911x_1 - 0.10368x_2 + 0.16993x_3 - 0.18953x_4 - 0.66777x_5 + 0.16754x_6$$
$$+0.06509x_7 - 0.40379x_8 - 0.27420x_9 - 0.33367x_{10} \geq 0\,,$$
$$p_{14}^2 - 0.00988x_1 - 0.19901x_2 - 0.41756x_3 - 0.37070x_4 + 0.59362x_5 + 0.36001x_6$$
$$-0.07059x_7 - 0.45195x_8 + 0.13747x_9 + 0.07904x_{10} \geq 0\,,$$
$$p_{15}^2 + 0.03658x_1 + 0.39325x_2 - 0.48337x_3 + 0.16317x_4 - 0.32896x_5 - 0.29999x_6$$
$$-0.18092x_7 - 0.20324x_8 - 0.37220x_9 + 0.01000x_{10} \geq 0\,,$$
$$p_{16}^2 - 0.21945x_1 - 0.01335x_2 - 0.16373x_3 + 0.08413x_4 - 0.06577x_5 + 0.07054x_6$$
$$+0.18376x_7 + 0.21854x_8 - 0.00187x_9 + 0.05499x_{10} \geq 0\,,$$
$$p_{17}^2 - 0.14342x_1 - 0.09385x_2 - 0.18046x_3 - 0.45135x_4 + 0.06394x_5 - 0.03482x_6$$
$$-0.01318x_7 - 0.17647x_8 - 0.21349x_9 - 0.49516x_{10} \geq 0\,,$$
$$p_{18}^2 + 0.56422x_1 + 0.10398x_2 + 0.15627x_3 - 0.08120x_4 + 0.17690x_5 + 0.26121x_6$$
$$+0.48809x_7 + 0.33354x_8 - 0.36254x_9 + 0.21566x_{10} \geq 0\,,$$
$$p_{19}^2 - 0.12633x_1 + 0.14551x_2 + 0.43341x_3 + 0.13317x_4 + 0.33814x_5 + 0.13453x_6$$
$$+0.03618x_7 - 0.18324x_8 - 0.17736x_9 - 0.02548x_{10} \geq 0\,,$$
$$p_{20}^2 - 0.17633x_1 + 0.22744x_2 - 0.20498x_3 - 0.50264x_4 - 0.21993x_5 - 0.08708x_6$$
$$+0.16424x_7 + 0.19095x_8 - 0.29382x_9 - 0.18516x_{10} \geq 0\,,$$
$$p_{21}^2 + 0.79170x_1 - 0.14420x_2 - 0.21757x_3 - 0.25966x_4 + 0.21661x_5 + 0.05421x_6$$
$$+0.02290x_7 - 0.50610x_8 - 0.13531x_9 - 0.22623x_{10} \geq 0\,,$$
$$p_{22}^2 - 0.50278x_1 - 0.51352x_2 - 0.53530x_3 - 0.08651x_4 - 0.18444x_5 - 0.51837x_6$$
$$-0.35092x_7 - 0.44002x_8 - 0.34156x_9 - 0.35032x_{10} \geq 0\,,$$
$$p_{23}^2 - 0.05378x_1 + 0.02732x_2 + 0.15293x_3 - 0.23920x_4 - 0.43243x_5 - 0.23813x_6$$
$$-0.17124x_7 - 0.01312x_8 - 0.39254x_9 - 0.20734x_{10} \geq 0\,,$$
$$p_{24}^2 + 0.07496x_1 + 0.14680x_2 - 0.03079x_3 + 0.17381x_4 - 0.19928x_5 + 0.04195x_6$$
$$-0.06350x_7 - 0.02453x_8 - 0.20124x_9 - 0.32000x_{10} \geq 0\,,$$
$$p_{25}^2 - 0.12078x_1 - 0.22002x_2 - 0.23673x_3 + 0.21513x_4 - 0.02143x_5 + 0.10087x_6$$
$$+0.44576x_7 + 0.35754x_8 + 0.33280x_9 - 0.63301x_{10} \geq 0\,,$$
$$p_{26}^2 + 0.12754x_1 + 0.14132x_2 + 0.69663x_3 - 0.08877x_4 - 0.03347x_5 - 0.13128x_6$$
$$+0.19682x_7 - 0.11034x_8 + 0.00844x_9 - 0.16935x_{10} \geq 0\,,$$
$$p_{27}^2 + 0.10432x_1 - 0.05707x_2 - 0.22885x_3 - 0.31006x_4 - 0.04702x_5 - 0.00418x_6$$
$$-0.39898x_7 - 0.12711x_8 - 0.20736x_9 + 1.23065x_{10} \geq 0\,,$$
$$p_{28}^2 + 0.04322x_1 + 0.41132x_2 - 0.19873x_3 + 0.37447x_4 - 0.21543x_5 + 0.35587x_6$$
$$+0.11709x_7 + 0.14721x_8 + 0.04446x_9 + 0.19899x_{10} \geq 0\,,$$
$$p_{29}^2 - 0.24439x_1 - 0.46030x_2 - 0.25595x_3 - 0.20200x_4 - 0.87509x_5 - 0.46547x_6$$
$$-0.50414x_7 - 0.52518x_8 - 0.41414x_9 - 0.66871x_{10} \geq 0\,,$$

$$
\begin{aligned}
& p_{30}^2 + 0.31355x_1 + 0.36232x_2 + 0.26860x_3 + 0.31180x_4 + 0.29723x_5 + 0.32587x_6 \\
& \quad +0.30609x_7 + 0.24988x_8 + 0.00513x_9 + 0.39233x_{10} \geq 0\,, \\
& p_{31}^2 - 0.21794x_1 - 0.16578x_2 - 0.09756x_3 - 0.25264x_4 - 0.17573x_5 - 0.11934x_6 \\
& \quad -0.01124x_7 + 0.09611x_8 - 0.07382x_9 - 0.10129x_{10} \geq 0\,, \\
& p_{32}^2 - 0.00200x_1 + 0.71206x_2 + 0.12424x_3 + 0.19938x_4 + 0.10814x_5 - 0.19292x_6 \\
& \quad -0.28982x_7 + 0.44038x_8 - 0.03806x_9 + 0.26840x_{10} \geq 0\,, \\
& p_{33}^2 - 0.16064x_1 - 0.15190x_2 + 0.66832x_3 + 0.44521x_4 - 0.04227x_5 - 0.33258x_6 \\
& \quad -0.23088x_7 + 0.65428x_8 - 0.31256x_9 + 0.32140x_{10} \geq 0\,, \\
& p_{34}^2 + 0.26690x_1 + 0.37228x_2 + 0.00857x_3 - 0.06232x_4 + 0.51136x_5 + 0.39388x_6 \\
& \quad +0.04392x_7 + 0.20031x_8 + 0.08393x_9 + 0.29775x_{10} \geq 0\,, \\
& p_{35}^2 + 0.03422x_1 + 0.49298x_2 + 0.70760x_3 + 0.08413x_4 - 0.43443x_5 - 0.42979x_6 \\
& \quad +0.10909x_7 + 0.08121x_8 + 0.56946x_9 - 0.42167x_{10} \geq 0\,, \\
& p_{36}^2 - 1.07149x_1 - 0.99997x_2 - 1.04208x_3 - 0.51296x_4 - 0.55606x_5 - 0.71676x_6 \\
& \quad -0.41995x_7 - 0.83098x_8 - 0.47672x_9 - 0.40903x_{10} \geq 0\,, \\
& 0.17499x_1 + 0.14240x_2 + 0.17240x_3 + 0.24587x_4 + 0.15108x_5 + 0.05292x_6 \\
& \qquad +0.10273x_7 + 0.16679x_8 + 0.29220x_9 + 0.41967x_{10} \geq 0.165\,, \\
& 0.19278x_1 + 0.14868x_2 + 0.15058x_3 + 0.17354x_4 + 0.27477x_5 + 0.11579x_6 \\
& \qquad +0.12072x_7 + 0.06188x_8 + 0.16188x_9 + 0.30120x_{10} \geq 0.195\,, \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,, \\
& y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} = 5\,, \\
& x_1 - 0.08y_1 \geq 0, x_2 - 0.08y_2 \geq 0, x_3 - 0.08y_3 \geq 0, x_4 - 0.08y_4 \geq 0\,, \\
& x_5 - 0.08y_5 \geq 0, x_6 - 0.08y_6 \geq 0, x_7 - 0.08y_7 \geq 0, x_8 - 0.08y_8 \geq 0\,, \\
& x_9 - 0.08y_9 \geq 0, x_{10} - 0.08y_{10} \geq 0\,, \\
& x_1 - 0.3y_1 \leq 0, x_2 - 0.3y_2 \leq 0, x_3 - 0.3y_3 \leq 0, x_4 - 0.3y_4 \leq 0\,, \\
& x_5 - 0.3y_5 \leq 0, x_6 - 0.3y_6 \leq 0, x_7 - 0.3y_7 \leq 0, x_8 - 0.3y_8 \leq 0\,, \\
& x_9 - 0.3y_9 \leq 0, x_{10} - 0.3y_{10} \leq 0\,, \\
& x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,, \\
& y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 10\,, \\
& p_t^2 \geq 0\,, \quad t = 1, 2, \ldots, 36\,.
\end{aligned}
$$

Now, in order to obtain portfolio selection strategy, we solve the above problem using the LINGO 12.0. The computational results summarized in Table 2.4 are based on four different sets of values of α, β and γ indicating the investor preferences in respect of return, risk and liquidity, respectively. Table 2.5 presents proportions of the assets in the obtained portfolios. Note that the intervals for return, risk and liquidity are constructed using equations (2.1)-(2.2) at the obtained solutions.

Table 2.4 Summary results of portfolio selection for conservative strategy

	Return	Risk	Liquidity
Portfolio 1	[0.19913, 0.24879]	[0.13183, 0.16554]	[0.00089, 0.00162]
Portfolio 2	[0.20412, 0.26434]	[0.13212, 0.16018]	[0.00081, 0.00122]
Portfolio 3	[0.18405, 0.23126]	[0.12327, 0.14952]	[0.00083, 0.00124]
Portfolio 4	[0.18601, 0.22713]	[0.12792, 0.15092]	[0.00101, 0.00188]

Table 2.5 The proportions of the assets in the obtained portfolio using conservative strategy

				Company				
	α	β	γ	ABL	ACL	ALL	BAL	CSL
Portfolio 1	1/3	1/3	1/3	0.3	0.24	0.0	0.08	0.0
Portfolio 2	2/3	1/2	1/3	0.3	0.08	0.0	0.24	0.0
Portfolio 3	1/3	2/3	1/3	0.3	0.08	0.0	0.08	0.0
Portfolio 4	1/3	1/2	2/3	0.3	0.3	0.0	0.08	0.0
				GIL	HBL	HUL	RIL	VOL
				0	0.08	0.0	0.0	0.3
				0	0.08	0.0	0.0	0.3
				0	0.28402	0.0	0.0	0.25598
				0	0.09810	0.0	0.0	0.2219

- **Aggressive strategy**

Using the input data from Tables 2.1-2.3, $r_{st} = 0.165$, $r_{lt} = 0.195$ and $h = 5$, we obtain portfolio selection strategy by solving the problem P(2.5) using the LINGO 12.0. The computational results summarized in Table 2.6 are based on four different sets of values of α, β and γ indicating the investor preferences in respect of return, risk and liquidity, respectively. Table 2.7 presents proportions of the assets in the obtained portfolios.

Table 2.6 Summary results of portfolio selection for aggressive strategy

	Return	Risk	Liquidity
Portfolio 1	[0.20489, 0.31122]	[0.13448, 0.19360]	[0.00301, 0.00329]
Portfolio 2	[0.20306, 0.31460]	[0.13576, 0.19725]	[0.00324, 0.00350]
Portfolio 3	[0.20088, 0.30425]	[0.11740, 0.16759]	[0.00290, 0.00323]
Portfolio 4	[0.15560, 0.24949]	[0.13018, 0.18797]	[0.00366, 0.00399]

Table 2.7 The proportions of the assets in the obtained portfolio using aggressive strategy

	α	β	γ	Company				
				ABL	ACL	ALL	BAL	CSL
Portfolio 1	1/3	1/3	1/3	0.1	0.0	0.0	0.14	0.16
Portfolio 2	2/3	1/3	1/3	0.08	0.0	0.0	0.08	0.24
Portfolio 3	1/3	2/3	1/3	0.0	0.0	0.0	0.24	0.08
Portfolio 4	1/10	1/2	2/3	0.0	0.0	0.0	0.08	0.28087
				GIL	HBL	HUL	RIL	VOL
				0.0	0.0	0.0	0.3	0.3
				0.0	0.0	0.0	0.3	0.3
				0.0	0.08	0.0	0.3	0.3
				0.0	0.25913	0.0	0.3	0.08

• **Combination strategy**

Using the input data from Tables 2.1-2.3, $r_{st} = 0.165$, $r_{lt} = 0.195$, $h = 5$ and $\alpha = \beta = \gamma = 1/3$, we obtain portfolio selection strategy by solving the problem P(2.6) using the LINGO 12.0. The computational results summarized in Table 2.8 are based on four different values of λ indicating the level of pessimism in the preferences of the investor. Table 2.9 presents proportions of the assets in the obtained portfolios.

Table 2.8 Summary results of portfolio selection for combination strategy

	λ	Return	Risk	Liquidity
Portfolio 1	0.1	[0.20306, 0.31460]	[0.13576, 0.19725]	[0.00324, 0.00350]
Portfolio 2	0.5	[0.20666, 0.30997]	[0.13303, 0.19093]	[0.00282, 0.00313]
Portfolio 3	0.7	[0.20876, 0.28120]	[0.13368, 0.17440]	[0.00132, 0.00175]
Portfolio 4	0.9	[0.20412, 0.26434]	[0.13183, 0.16554]	[0.00081, 0.00122]

A comparison of the solutions listed in Tables 2.4, 2.6 and 2.8 highlights that the portfolio selection models discussed in this chapter are capable not only in capturing the investor attitudes among investor types, but also in capturing individual preferences among the investors within a given type.

Table 2.9 The proportions of the assets in the obtained portfolio using combination strategy

	Company				
	ABL	ACL	ALL	BAL	CSL
Portfolio 1	0.08	0.0	0.0	0.08	0.24
Portfolio 2	0.08	0.0	0.0	0.24	0.08
Portfolio 3	0.24	0.08	0.0	0.3	0.0
Portfolio 4	0.3	0.08	0.0	0.24	0.0
	GIL	HBL	HUL	RIL	VOL
	0.0	0.0	0.0	0.3	0.3
	0.0	0.0	0.0	0.3	0.3
	0.0	0.0	0.0	0.08	0.3
	0.0	0.08	0.0	0.0	0.3

2.5 Comments

In this chapter, we have presented the following facts:

- In practise one needs to consider currently available information as well as future scenarios for estimating return, risk and liquidity of assets. A well constructed interval can be a good measure of uncertainty associated with the behavior of asset prices.
- A framework of portfolio selection using the concept of interval coefficients has been discussed.
- To capture investor attitude, three different models of portfolio selection corresponding to conservative strategy, aggressive strategy and combination strategy have been discussed.
- Moreover, it has been shown that the portfolio selection models can generate satisfying portfolios using intervals that represent degrees of optimism and pessimism in respect of portfolio parameters by performing numerical experiments based on real-world data.

Chapter 3
Portfolio Optimization in Fuzzy Environment

Abstract. In this chapter, we discuss a bi-objective fuzzy portfolio selection model that maximizes the portfolio return and minimizes the portfolio risk. We use an fuzzy interactive approach to solve the model so that the desired aspiration levels of the decision maker with regard to return and risk objectives are achieved as closely as possible.

3.1 Fuzzy Decision Theory

The investor expectations regarding financial parameters on the basis of which portfolio decisions are taken are often vaguely stated, e.g. '*one would expect a return significantly more than 30%*' or '*one would expect a risk significantly lower than 15%*'. Constructing satisfactory portfolios on the basis of such vague expressions poses a methodological challenge that can not be met using crisp numbers or even the interval numbers. Under such situations recourse to fuzzy set theory would be more useful; thus, there is a growing reliance on fuzzy set theory for modeling portfolio selection problems. Fuzzy set theory not only captures the vagueness and uncertainty but also provides the flexibility in decision making by integrating the subjective preferences of the investors and knowledge of the experts.

Zadeh [123] introduced the concept of fuzzy sets in 1965. Further, based on this concept, Bellman and Zadeh [7] presented fuzzy decision theory. They defined decision making in a fuzzy environment with a decision set which unifies fuzzy goals and fuzzy constraints. In fuzzy set theory, unlike the classical set theory, there is no sharp boundary between those elements that belong to the set and those that do not.

Definition 3.1 (Fuzzy set). Let X be a universe whose generic element be denoted by x. A fuzzy set A in X is a set of ordered pairs $A = \{(x, \mu_A(x)) : x \in X\}$, where $\mu_A(x)$ is the membership function or grade of membership of $x \in X$ defined on the real interval [0,1].

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Studies in Fuzziness and Soft Computing 316,
DOI: 10.1007/978-3-642-54652-5_3,

The fuzzy set A in X is thus uniquely characterized by its membership function $\mu_A(x)$, which associates with each element in X, a non-negative real number whose value is finite and lies in the interval $[0,1]$. The value of $\mu_A(x)$ at x represents the 'grade of membership' of x in A. Thus, nearer the value of $\mu_A(x)$ to 1, higher the grade of 'belongingness' of x in A.

Example 3.1. *Let* $X = \{2000, 4000, 6000, 8000\}$ *be the set of possible amounts that an individual investor desires to invest in the financial market. Then the fuzzy set A of 'comfortable investments' may be defined subjectively by an investor according to his/her risk bearing capacity as*

$$\mu(x = 2000) = 1, \mu(x = 4000) = 0.8, \mu(x = 6000) = 0.6, \mu(x = 8000) = 0.4$$

where $\mu(\cdot)$ *is the membership function of the fuzzy set A of X. The fuzzy set can also be represented as* $A = \{(2000,1), (4000,0.8), (6000,0.6), (8000,0.4)\}$.

Further, suppose that fuzzy sets are defined on a set of alternatives, X, to a decision problem. Then a fuzzy goal G in X may be identified with a given fuzzy set G in X. Similarly, one can define a fuzzy constraint C in X. Given the fuzzy goals and fuzzy constraints, one can define the decision making situation in a fuzzy environment as the intersection of goals and constraints. It may be noted that in a fuzzy environment the goals and constraints are treated similarly. More specifically, if we are given a space of decision alternatives X, then the fuzzy decision D be defined as a fuzzy set in X given by $D = G \cap C$ where $\cap$ is a conjunctive operator, which has different alternatives and different meanings in practical situations. In terms of the membership functions, the fuzzy decision can be formulated as

$$\mu_D(x) = \min(\mu_G(x), \mu_C(x)), \quad \forall\ x \in X,$$

where $\mu_G(x)$ and $\mu_C(x)$ are the membership functions of the fuzzy goal and the fuzzy constraint, respectively. More generally, if there are m fuzzy goals $G_i (i = 1, \ldots, m)$ and n fuzzy constraints $C_j (j = 1, \ldots, n)$, then the fuzzy decision is defined by the following fuzzy set

$$D = \{G_1 \cap G_2 \cap \ldots \cap G_m\} \cap \{C_1 \cap C_2 \cap \ldots \cap C_n\}$$

and its membership function is characterized as

$$\mu_D(x) = \min(\mu_{G_1}(x), \mu_{G_2}(x), \ldots, \mu_{G_m}(x), \mu_{C_1}(x), \mu_{C_2}(x), \ldots, \mu_{C_n}(x)), \quad \forall\ x \in X.$$

Bellman and Zadeh [7] proposed a maximizing decision x^* defined by the following non-fuzzy set

$$D^* = \{x^* \in X | x^* = \text{argmax}\{\mu_D(x)\} = \text{argmax}\{\min(\mu_G(x), \mu_C(x))\}\}.$$

More specifically, when m fuzzy objectives and n fuzzy constraints are given, the optimal decision may be obtained as follows:

$$D^* = \{x^* \in X | x^* = \operatorname{argmax}\{\mu_D(x)\}$$
$$= \operatorname{argmax}\{\min(\mu_{G_1}(x), \mu_{G_2}(x), \ldots, \mu_{G_m}(x), \mu_{C_1}(x), \mu_{C_2}(x), \ldots, \mu_{C_n}(x))\}\}.$$

In other words, the maximizing decision x^* may be defined as an alternative with the highest membership in the fuzzy decision D, i.e.,

$$\mu_D(x^*) = \bigcup_{x \in X} \mu_D(x).$$

The maximizing decision x^* may be considered to be an optimal decision in a sense that it can be interpreted in different ways, depending on the definitions of the operators $\cap$ and $\cup$. The operator $\cap$ may be extended to various forms of conjunctive operators such as minimum operator, weighted sum of the goals and the constraints, multiplication operator, mean value operator, bounded product, Hamacher's min operator. The operator $\cup$ can be substituted by algebraic sum, bounded sum, Yager's max operator. A detailed discussion on these operators is presented in [74]. Among these operators, the max-min operator, i.e., Bellman-Zadeh's approach [7] is commonly used in practice. The selection of the operators depend on the preferences of the decision maker, the problem-context and semantic interpretation.

Example 3.2. *Let us assume that the investor sets the goal for a portfolio return performance indicator* $r(x)$ *to 'significantly more than a level* $r_0 = 15\%$*'. This vague goal in respect of portfolio return can be modeled by means of its linear membership function [128, 129] as follows:*

$$\mu_r(x) = \begin{cases} 1, & \text{if } r(x) \geq r_1, \\ \frac{r(x) - r_0}{r_1 - r_0}, & \text{if } r_0 < r(x) < r_1, \\ 0, & \text{if } r(x) \leq r_0, \end{cases}$$

where $r_0 = 15\%$ *is the lower aspiration level and* $r_1 = 30\%$ *is the upper aspiration level of the investor in respect of portfolio return. The membership function is shown in Fig. 3.1. The* x*-axis represents all possible values for* $r(x)$ *while the* y*-axis measures the overall degree of goal attainment. The investor effectively rejects solutions for which the value of* $r(x)$ *falls below* r_0 *that corresponds to degree of satisfaction equal to* 0*. As* $r(x)$ *increases, so does the goal fulfilment and the investor is practically indifferent towards any solution for which the value of* $r(x)$ *exceeds* r_1*. The portfolio returns between* r_0 *and* r_1 *are acceptable at the varying degrees of satisfaction between 0 and 1.*

In a similar manner, one can formulate goal or constraint for a portfolio risk performance indicator $w(x)$ *to 'significantly less than a level* $w_1 = 40\%$*'. This vague constraint in respect of portfolio risk can be modeled by means of its membership function as follows:*

$$\mu_w(x) = \begin{cases} 1, & \text{if } w(x) \leq w_0, \\ \frac{w_1 - w(x)}{w_1 - w_0}, & \text{if } w_0 < w(x) < w_1, \\ 0, & \text{if } w(x) \geq w_1, \end{cases}$$

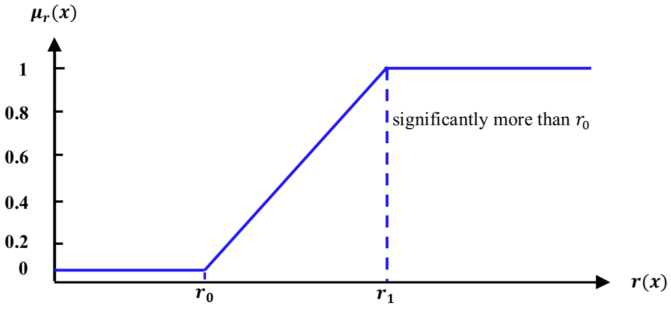

Fig. 3.1 Graphical representation of membership function of portfolio return

where $w_1 = 40\%$ *is the upper aspiration level and* $w_0 = 20\%$ *is the lower aspiration level of the investor in respect of portfolio risk. The membership function is shown in Fig. 3.2. The investor effectively rejects solutions for which the value of* $w(x)$ *falls above* w_1 *corresponding to degree of satisfaction equal to* 0*. As* $w(x)$ *decreases, the goal fulfilment increases and the investor is practically indifferent towards any solution for which the value of* $w(x)$ *is less than* w_0*. The portfolio risks between* w_0 *and* w_1 *are acceptable at the varying degrees of satisfaction between 0 and 1.*

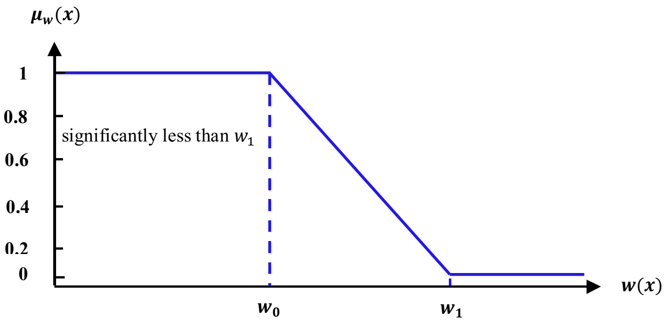

Fig. 3.2 Graphical representation of membership function of portfolio risk

Now, using Bellman-Zadeh's approach [7], the fuzzy decision D equals $r(x) \cap w(x)$*, i.e.,*

$$\mu_D(x) = \min(\mu_r(x), \mu_w(x)).$$

Fig. 3.3 depicts the fuzzy decision situation and identifies the optimal solution x^* *corresponding to which portfolio return is* r^* *and portfolio risk is* w^**.*

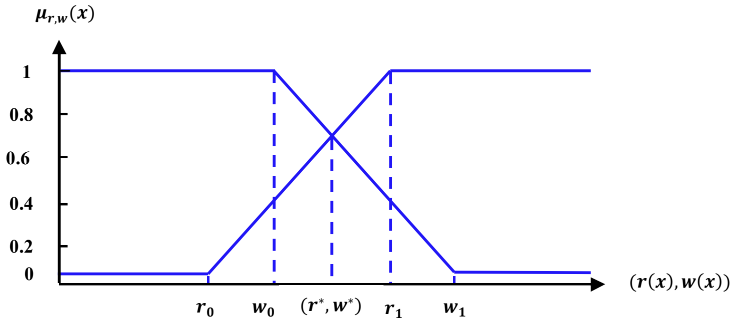

Fig. 3.3 Graphical representation of fuzzy decision

3.2 Fuzzy Portfolio Selection Model

We consider a bi-objective portfolio optimization model based on mean-variance framework proposed by Markowitz. The model simultaneously maximize the portfolio return ($f_1(x)$) and minimize the portfolio risk ($f_2(x)$) and is formulated as follows:

$$\textbf{P(3.1)} \quad \max \; f_1(x) = \sum_{i=1}^{n} r_i x_i$$

$$\min \; f_2(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_i x_j$$

subject to

$$\sum_{i=1}^{n} x_i = 1\,, \tag{3.1}$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,, \tag{3.2}$$

where $r_i = E[R_i]$, $\sigma_{ij} = E[(R_i - r_i)(R_j - r_j)]$, i.e., the covariance between assets i and j. The problem P(3.1) is a quadratic programming problem.

Considering that in the real-world applications of portfolio selection, decisions are often structured around vague aspirations about the desired portfolio return and risk, we present a fuzzy framework for accommodating approximate linguistic-type information in the portfolio selection problem. Some of the relevant references for fuzzy framework of portfolio selection using mean-variance model are [9, 100, 101, 119, 127]. We formulate a fuzzy bi-objective portfolio selection problem based on vague aspiration levels of investors regarding portfolio return and risk to determine a satisfying portfolio selection strategy. It is assumed that investors indicate aspiration levels on the basis of their prior experience and knowledge and linear membership functions are used to express such vague aspiration levels of the investors. A linear membership function is most commonly used because it is simple and it is defined by fixing two points: the upper and lower levels of acceptability (as defined in Section 3.1 for fuzzy goal and fuzzy constraint).

The linear membership function of the goal of expected portfolio return is defined as follows:

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq f_1^R, \\ \frac{f_1(x) - f_1^L}{f_1^R - f_1^L}, & \text{if } f_1^L < f_1(x) < f_1^R, \\ 0, & \text{if } f_1(x) \leq f_1^L, \end{cases}$$

where f_1^L is the worst lower bound (lower aspiration level) and f_1^R is the best upper bound (upper aspiration level) of the portfolio return, see Fig. 3.4.

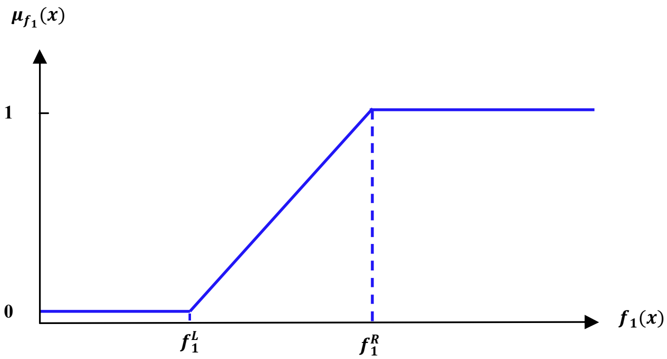

Fig. 3.4 Graphical representation of goal of fuzzy portfolio return

Similarly, the membership function of the goal of portfolio risk is given by

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \leq f_2^L, \\ \frac{f_2^R - f_2(x)}{f_2^R - f_2^L}, & \text{if } f_2^L < f_2(x) < f_2^R, \\ 0, & \text{if } f_2(x) \geq f_2^R, \end{cases}$$

where f_2^R is the worst upper bound (upper aspiration level) and f_2^L is the best lower bound (lower aspiration level) of the portfolio risk, see Fig. 3.5.

Following Bellman-Zadeh's maximization approach [7] and using the above defined fuzzy membership functions, the fuzzy bi-objective optimization model for portfolio selection problem is formulated as follows:

$$
\begin{aligned}
\mathbf{P(3.2)} \quad & \max \lambda \\
& \text{subject to} \\
& \lambda \le \mu_{f_1}(x)\,, \\
& \lambda \le \mu_{f_2}(x)\,, \\
& \sum_{i=1}^{n} x_i = 1\,, \\
& x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,, \\
& 0 \le \lambda \le 1\,,
\end{aligned}
$$

where λ is an auxiliary variable representing the grade of membership.

In real-world multiobjective portfolio selection problems (like problem P(3.2)), the investor desires for a solution which is as close to his/her expectations as possible in terms of attainment level of the objectives. To find such a '*compromise solution*' we need to solve the model P(3.2) iteratively in an interactive manner wherein the investor is initially required to specify the preferences and expectations. Based on these inputs the problem is solved and the investor is provided with a possible solution. If the investor is satisfied with the obtained solution, it is considered as '*preferred compromise solution*' and the process terminates; otherwise, the preferences need to be modified in the light of the results obtained. This iterative procedure is continued till a preferred compromise solution is obtained.

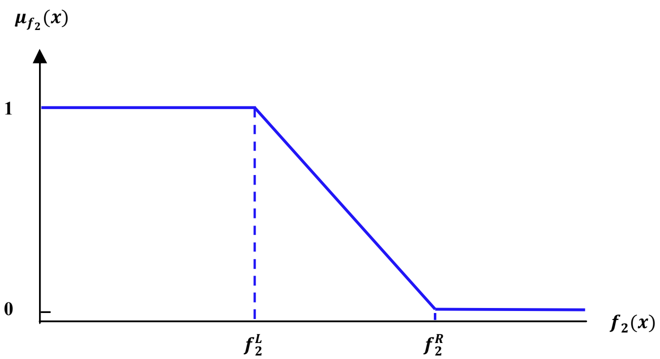

Fig. 3.5 Graphical representation of goal of fuzzy portfolio risk

3.3 Solution Methodology

We now present a fuzzy interactive approach based on the idea discussed in [1] to incorporate investor preferences for the attainment level of the objective functions. This interactive approach follows a number of iterations that might be necessary for attaining the preferred compromise solution. In the initial stage of determining the bounds (aspiration levels) in respect of the objective functions, the involvement of the investor is not necessary. If the aspiration levels are too tight, then there is no achievable solution and the investor is required to relax the aspiration levels. On the other hand, if the aspiration levels are too weak, then too many solutions may be generated and it becomes difficult for the investor to identify the preferred compromise solution. To determine the appropriate bounds, we solve the problem P(3.1) as a single-objective problem in respect of return and risk objective functions. The worst lower (upper) bounds and best upper (lower) bounds are obtained by evaluating the objective functions at both the solutions. These bounds are used to construct the membership functions for the problem P(3.2) and the obtained solution is provided to the investor. If the investor is satisfied, the obtained solution is considered as the preferred compromise solution and the solution process terminates. Otherwise, both the objective functions are re-evaluated and the worst bound(s) are modified for further possible improvement in the attainment level of the objective(s). This approach has the following advantages.

(i) It helps in minimizing the information needed from the investor.
(ii) It does not require the investor to specify bounds (aspiration levels) in respect of the objective functions.
(iii) It helps in reducing the number of iterations in the interactive approach to reach the preferred compromise solution.
(iv) The investor has greater confidence in the solution obtained.

Stepwise Description of the Fuzzy Interactive Approach

The solution methodology of the fuzzy interactive approach for problem P(3.2) consists of the following steps:

Step 1: Construct the mathematical model P(3.1).

Step 2: Solve the problem P(3.1) as a single-objective problem in respect of return and risk objective functions. Mathematically,

For return objective function:

$$\max f_1(x), \text{ subject to constraints (3.1)-(3.2).}$$

For risk objective function:

$$\min f_2(x), \text{ subject to constraints (3.1)-(3.2).}$$

Let x^1 and x^2 denote the optimal solutions obtained by solving the single-objective problems in respect of return and risk objective functions, respectively. If both the solutions, i.e., $x^1 = x^2 = (x_1, x_2, \ldots, x_n)$ are same, we obtain an efficient (preferred compromise) solution and stop; otherwise, go to Step 3.

Step 3: Evaluate the objective functions at the obtained solutions. Determine the worst lower bound (f_1^L) and best upper bound (f_1^R) for return objective; and, the best lower bound (f_2^L) and worst upper bound (f_2^R) for risk objective. We obtain these bounds as

$$f_1^R = f_1(x^1),$$

$$f_1^L = f_1(x^2),$$

$$f_2^L = f_2(x^2),$$

$$f_2^R = f_2(x^1).$$

Step 4: Define the linear membership functions for return and risk.

Step 5: Develop the mathematical model P(3.2) and solve it. Present the solution to the investor. If the investor accepts it then stop; otherwise, re-evaluate both the objective functions. For the return objective, compare the present worst lower bound with the new objective value. If the new value is higher than the worst lower bound, consider it as a new lower bound; otherwise, use the old value as is. On the other hand, for the risk objective, compare the present worst upper bound with the new objective value. If the new value is lower than the worst upper bound, consider it as a new upper bound; otherwise, use the old value as is. If there are no changes in current bounds of both the objective functions then stop; otherwise go to Step 4 and re-iterate the solution process.

Fig. 3.6 depicts the flowchart of the fuzzy interactive approach.

Remark 3.1. *The preferred compromise solution is the optimal solution of problem P(3.2) and consequently this solution is an efficient solution (for more details about this readers may refer to [78] and [113]).*

3.4 Numerical Illustration

The model presented in Section 3.2 is empirically verified in this section with reference to the data set extracted from NSE, Mumbai, India corresponding to ten assets. Table 3.1 provides the data corresponding to expected return. Table 3.2 provides the data corresponding to variance and covariance among the 10 randomly selected assets.

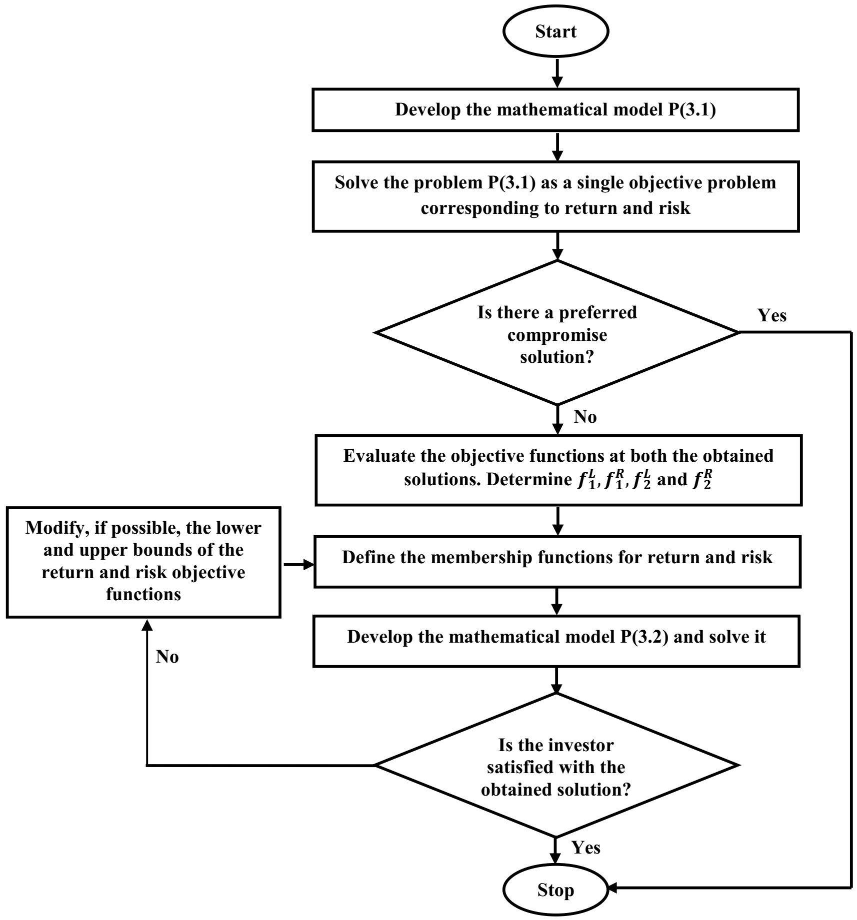

Fig. 3.6 Flow chart of the fuzzy interactive approach

Table 3.1 Input data of assets corresponding to expected return

Company	Return
A B B Ltd. (ABL)	0.19278
Alfa Laval (India) Ltd. (ALL)	0.13587
Bajaj Hindusthan Ltd. (BHL)	0.40086
Crompton Greaves Ltd. (CGL)	0.29892
Hero Honda Motors Ltd. (HHM)	0.14921
Hindustan Construction Co. Ltd. (HCC)	0.30107
Kotak Mahindra Bank Ltd. (KMB)	0.23818
Mahindra & Mahindra Ltd. (MML)	0.23114
Siemens Ltd. (SIL)	0.26122
Unitech Ltd. (UNL)	0.56246

Table 3.2 Input data corresponding to variance and covariance among assets

Company	ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL
ABL	0.14010	0.05039	0.09355	0.09625	0.04316	0.06345	0.06605	0.03460	0.10983	0.01299
ALL	0.05039	0.08682	0.10359	0.07729	0.02587	0.07375	0.04384	0.01895	0.07371	0.05102
BHL	0.09355	0.10359	0.42326	0.17952	0.04742	0.13237	0.02961	0.05984	0.15413	0.04477
CGL	0.09625	0.07729	0.17952	0.23027	0.02616	0.09370	0.01250	0.00350	0.11755	0.04725
HHM	0.04316	0.02587	0.04742	0.02616	0.07587	0.03010	0.05828	0.06190	0.05702	-0.01761
HCC	0.06345	0.07375	0.13237	0.09370	0.03010	0.18957	0.04537	0.02684	0.10445	-0.00957
KMB	0.06605	0.04384	0.02961	0.01250	0.05828	0.04537	0.16280	0.09316	0.07749	0.07029
MML	0.03460	0.01895	0.05984	0.00350	0.06190	0.02684	0.09316	0.11891	0.05333	-0.01455
SIL	0.10983	0.07371	0.15413	0.11755	0.05702	0.10445	0.07749	0.05333	0.18988	0.06564
UNL	0.01299	0.05102	0.04477	0.04725	-0.01761	-0.00957	0.07029	-0.01455	0.06564	0.58191

Portfolio Selection

In order to find an optimal asset allocation (i.e., preferred compromise solution), we use the interactive approach discussed in Section 3.3.

Step 1: We formulate the model P(3.1) using the input data from Tables 3.1-3.2 as follows:

$$
\begin{aligned}
\max\ f_1(x) = {} & 0.19278x_1 + 0.13587x_2 + 0.40086x_3 + 0.29892x_4 + 0.14921x_5 \\
& +0.30107x_6 + 0.23818x_7 + 0.23114x_8 + 0.26122x_9 + 0.56246x_{10} \\
\min\ f_2(x) = {} & 0.14010x_1x_1 + 0.08682x_2x_2 + 0.42326x_3x_3 + 0.23027x_4x_4 \\
& +0.07587x_5x_5 + 0.18957x_6x_6 + 0.16280x_7x_7 + 0.11891x_8x_8 \\
& +0.18988x_9x_9 + 0.58191x_{10}x_{10} + 0.10078x_1x_2 + 0.18711x_1x_3 \\
& +0.19250x_1x_4 + 0.08633x_1x_5 + 0.12689x_1x_6 + 0.13209x_1x_7 \\
& +0.06919x_1x_8 + 0.21966x_1x_9 + 0.02598x_1x_{10} + 0.20718x_2x_3 \\
& +0.15458x_2x_4 + 0.05175x_2x_5 + 0.14749x_2x_6 + 0.08768x_2x_7 \\
& +0.03790x_2x_8 + 0.14741x_2x_9 + 0.10203x_2x_{10} + 0.35904x_3x_4 \\
& +0.09484x_3x_5 + 0.26475x_3x_6 + 0.05923x_3x_7 + 0.11967x_3x_8 \\
& +0.30827x_3x_9 + 0.08954x_3x_{10} + 0.05233x_4x_5 + 0.18741x_4x_6 \\
& +0.02499x_4x_7 + 0.00701x_4x_8 + 0.23510x_4x_9 + 0.09450x_4x_{10} \\
& +0.06021x_5x_6 + 0.11656x_5x_7 + 0.12379x_5x_8 + 0.11404x_5x_9 \\
& -0.03522x_5x_{10} + 0.09074x_6x_7 + 0.05367x_6x_8 + 0.20890x_6x_9 \\
& -0.01913x_6x_{10} + 0.18631x_7x_8 + 0.15498x_7x_9 + 0.14058x_7x_{10} \\
& +0.10665x_8x_9 - 0.02910x_8x_{10} + 0.13128x_9x_{10}
\end{aligned}
$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,,$$
$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,.$$

Step 2: To determine the worst lower (upper) bound and best upper (lower) bound for return and risk objective functions, respectively, the above problem is solved as a single-objective problem as follows:

For Return Objective Function

$$
\begin{aligned}
\max\ f_1(x) = {} & 0.19278x_1 + 0.13587x_2 + 0.40086x_3 + 0.29892x_4 + 0.14921x_5 \\
& +0.30107x_6 + 0.23818x_7 + 0.23114x_8 + 0.26122x_9 + 0.56246x_{10}
\end{aligned}
$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,,$$
$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,.$$

The obtained solution (x^1) using the LINGO 12.0 is provided in Table 3.3.

For Risk Objective Function

$$
\begin{aligned}
\min \ f_2(x) = {} & 0.14010x_1x_1 + 0.08682x_2x_2 + 0.42326x_3x_3 + 0.23027x_4x_4 \\
& +0.07587x_5x_5 + 0.18957x_6x_6 + 0.16280x_7x_7 + 0.11891x_8x_8 \\
& +0.18988x_9x_9 + 0.58191x_{10}x_{10} + 0.10078x_1x_2 + 0.18711x_1x_3 \\
& +0.19250x_1x_4 + 0.08633x_1x_5 + 0.12689x_1x_6 + 0.13209x_1x_7 \\
& +0.06919x_1x_8 + 0.21966x_1x_9 + 0.02598x_1x_{10} + 0.20718x_2x_3 \\
& +0.15458x_2x_4 + 0.05175x_2x_5 + 0.14749x_2x_6 + 0.08768x_2x_7 \\
& +0.03790x_2x_8 + 0.14741x_2x_9 + 0.10203x_2x_{10} + 0.35904x_3x_4 \\
& +0.09484x_3x_5 + 0.26475x_3x_6 + 0.05923x_3x_7 + 0.11967x_3x_8 \\
& +0.30827x_3x_9 + 0.08954x_3x_{10} + 0.05233x_4x_5 + 0.18741x_4x_6 \\
& +0.02499x_4x_7 + 0.00701x_4x_8 + 0.23510x_4x_9 + 0.09450x_4x_{10} \\
& +0.06021x_5x_6 + 0.11656x_5x_7 + 0.12379x_5x_8 + 0.11404x_5x_9 \\
& -0.03522x_5x_{10} + 0.09074x_6x_7 + 0.05367x_6x_8 + 0.20890x_6x_9 \\
& -0.01913x_6x_{10} + 0.18631x_7x_8 + 0.15498x_7x_9 + 0.14058x_7x_{10} \\
& +0.10665x_8x_9 - 0.02910x_8x_{10} + 0.13128x_9x_{10}
\end{aligned}
$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,,$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,.$$

The obtained solution (x^2) is provided in Table 3.3.

Table 3.3 The proportions of the assets in the obtained portfolio corresponding to single-objectives

	Allocation				
	ABL	ALL	BHL	CGL	HHM
x^1	0.0	0.0	0.0	0.0	0.0
x^2	0.05151	0.29415	0.0	0.01321	0.38874
	HCC	KMB	MML	SIL	UNL
x^1	0.0	0.0	0.0	0.0	1.0
x^2	0.03884	0.0	0.14241	0.0	0.07114

Step 3: We evaluate both the objective functions at the obtained solutions, i.e., x^1 and x^2. The objective function values are provided in Table 3.4. Now, the worst lower (upper) bound and best upper (lower) bound of both the objective functions are obtained as follows:

$$0.19647 \leq f_1(x) \leq 0.56246$$

$$0.04841 \leq f_2(x) \leq 0.58191$$

Table 3.4 Objective function values of return and risk at both obtained solutions

	x^1	x^2
Return ($f_1(x)$)	0.56246	0.19647
Risk ($f_2(x)$)	0.58191	0.04841

Step 4: The membership functions for return and risk are constructed as follows:

The linear membership function of the objective of expected portfolio return is

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq 0.56246\,, \\ \frac{f_1(x)-0.19647}{0.56246-0.19647}, & \text{if } 0.19647 < f_1(x) < 0.56246\,, \\ 0, & \text{if } f_1(x) \leq 0.19647\,. \end{cases}$$

The linear membership function of the objective of portfolio risk is

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \leq 0.04841\,, \\ \frac{0.58191-f_2(x)}{0.58191-0.04841}, & \text{if } 0.04841 < f_2(x) < 0.58191\,, \\ 0, & \text{if } f_2(x) \geq 0.58191\,. \end{cases}$$

Step 5: We formulate the model P(3.2) as follows:

$$\max\ \lambda$$

subject to

$$\begin{aligned}
&0.19278x_1 + 0.13587x_2 + 0.40086x_3 + 0.29892x_4 + 0.14921x_5 + 0.30107x_6 \\
&\quad +0.23818x_7 + 0.23114x_8 + 0.26122x_9 + 0.56246x_{10} - 0.36599\lambda \geq 0.19647\,, \\
&0.14010x_1x_1 + 0.08682x_2x_2 + 0.42326x_3x_3 + 0.23027x_4x_4 \\
&\quad +0.07587x_5x_5 + 0.18957x_6x_6 + 0.16280x_7x_7 + 0.11891x_8x_8 + 0.18988x_9x_9 \\
&\quad +0.58191x_{10}x_{10} + 0.10078x_1x_2 + 0.18711x_1x_3 + 0.19250x_1x_4 + 0.08633x_1x_5 \\
&\quad +0.12689x_1x_6 + 0.13209x_1x_7 + 0.06919x_1x_8 + 0.21966x_1x_9 + 0.02598x_1x_{10} \\
&\quad +0.20718x_2x_3 + 0.15458x_2x_4 + 0.05175x_2x_5 + 0.14749x_2x_6 + 0.08768x_2x_7 \\
&\quad +0.03790x_2x_8 + 0.14741x_2x_9 + 0.10203x_2x_{10} + 0.35904x_3x_4 + 0.09484x_3x_5 \\
&\quad +0.26475x_3x_6 + 0.05923x_3x_7 + 0.11967x_3x_8 + 0.30827x_3x_9 + 0.08954x_3x_{10} \\
&\quad +0.05233x_4x_5 + 0.18741x_4x_6 + 0.02499x_4x_7 + 0.00701x_4x_8 + 0.23510x_4x_9 \\
&\quad +0.09450x_4x_{10} + 0.06021x_5x_6 + 0.11656x_5x_7 + 0.12379x_5x_8 + 0.11404x_5x_9 \\
&\quad -0.03522x_5x_{10} + 0.09074x_6x_7 + 0.05367x_6x_8 + 0.20890x_6x_9 - 0.01913x_6x_{10} \\
&\quad +0.18631x_7x_8 + 0.15498x_7x_9 + 0.14058x_7x_{10} + 0.10665x_8x_9 - 0.02910x_8x_{10} \\
&\quad +0.13128x_9x_{10} + 0.5335\lambda \leq 0.58191\,,
\end{aligned}$$

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,,$$
$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 10\,,$$
$$0 \leq \lambda \leq 1\,.$$

The above problem is solved using the LINGO 12.0 and the computational results are summarized in Table 3.5. Table 3.6 presents proportions of the assets in the obtained portfolio.

Table 3.5 Summary results of portfolio selection

λ	Return ($f_1(x)$)	Risk ($f_2(x)$)
0.70541	0.45464	0.20557

Table 3.6 The proportions of the assets in the obtained portfolio

	Allocation				
	ABL	ALL	BHL	CGL	HHM
Portfolio	0.0	0.0	0.24646	0.0	0.0
	HCC	KMB	MML	SIL	UNL
	0.24659	0.0	0.01067	0.0	0.49628

Now, suppose the investor is not satisfied with the solution obtained and wants to improve it further. As desired by the investor, an individual objective, i.e., return (risk) can be improved; however, due to the multiobjective nature of the problem, the improvement in one objective can produce adverse effects on other objective. Hence, depending upon investor preferences for both the objectives, we can modify the obtained solution. In this process, the lower (upper) bound and aspiration level of the selected objective function are modified. The fuzzy problem P(3.2) is resolved with the new parameters and this process is re-iterated until the investor terminates the process.

An important issue at this stage that needs to be addressed is whether the proposed model fit across investor types? In order to test this, let us consider two cases.

- **Case 1: Aggressive investor**

Suppose the investor is not satisfied with the obtained portfolio presented in Table 3.6. Since the investor follows aggressive strategy, the portfolio return is improved at the cost of taking a higher risk level. To do so, as the obtained portfolio return value (see Table 3.5) is higher than the worst lower bound, we consider it as a new lower bound and no changes are made in the bounds of the risk objective. The problem P(3.2) is solved with new parameters. Table 3.7 lists some sample preferred compromise solutions obtained by varying the lower bound of the return objective.

Table 3.7 The proportions of the assets in the obtained portfolios by varying the lower bound of the return objective

	Bounds	λ	Return	Risk	Allocation									
			$f_1(x)$	$f_2(x)$	ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL
Portfolio 1	$0.45464 \le f_1(x) \le 0.56246$	0.48774	0.50723	0.32170	0.0	0.0	0.34178	0.0	0.0	0.0	0.0	0.0	0.0	0.65822
Portfolio 2	$0.50723 \le f_1(x) \le 0.56246$	0.35884	0.52705	0.39047	0.0	0.0	0.21913	0.0	0.0	0.0	0.0	0.0	0.0	0.78087
Portfolio 3	$0.52705 \le f_1(x) \le 0.56246$	0.27620	0.53683	0.43456	0.0	0.0	0.15860	0.0	0.0	0.0	0.0	0.0	0.0	0.84140
Portfolio 4	$0.53683 \le f_1(x) \le 0.56246$	0.22227	0.54253	0.46333	0.0	0.0	0.12335	0.0	0.0	0.0	0.0	0.0	0.0	0.87665
Portfolio 5	$0.54253 \le f_1(x) \le 0.56246$	0.18505	0.54622	0.48318	0.0	0.0	0.10051	0.0	0.0	0.0	0.0	0.0	0.0	0.89949
Portfolio 6	$0.54622 \le f_1(x) \le 0.56246$	0.15809	0.54879	0.49757	0.0	0.0	0.08461	0.0	0.0	0.0	0.0	0.0	0.0	0.91539
Portfolio 7	$0.54879 \le f_1(x) \le 0.56246$	0.13774	0.55068	0.50842	0.0	0.0	0.07294	0.0	0.0	0.0	0.0	0.0	0.0	0.92706
Portfolio 8	$0.55068 \le f_1(x) \le 0.56246$	0.12187	0.55212	0.51690	0.0	0.0	0.06401	0.0	0.0	0.0	0.0	0.0	0.0	0.93599
Portfolio 9	$0.55212 \le f_1(x) \le 0.56246$	0.10920	0.55325	0.52365	0.0	0.0	0.05700	0.0	0.0	0.0	0.0	0.0	0.0	0.94300
Portfolio 10	$0.55325 \le f_1(x) \le 0.56246$	0.09889	0.55416	0.52915	0.0	0.0	0.05136	0.0	0.0	0.0	0.0	0.0	0.0	0.94864
Portfolio 11	$0.55416 \le f_1(x) \le 0.56246$	0.09033	0.55491	0.53372	0.0	0.0	0.04672	0.0	0.0	0.0	0.0	0.0	0.0	0.95328
Portfolio 12	$0.55491 \le f_1(x) \le 0.56246$	0.08311	0.55554	0.53757	0.0	0.0	0.04284	0.0	0.0	0.0	0.0	0.0	0.0	0.95716
Portfolio 13	$0.55554 \le f_1(x) \le 0.56246$	0.07692	0.55608	0.54087	0.0	0.0	0.03953	0.0	0.0	0.0	0.0	0.0	0.0	0.96047
Portfolio 14	$0.55608 \le f_1(x) \le 0.56246$	0.07151	0.55654	0.54376	0.0	0.0	0.03666	0.0	0.0	0.0	0.0	0.0	0.0	0.96334
Portfolio 15	$0.55654 \le f_1(x) \le 0.56246$	0.06683	0.55694	0.54625	0.0	0.0	0.03419	0.0	0.0	0.0	0.0	0.0	0.0	0.96581
Portfolio 16	$0.55694 \le f_1(x) \le 0.56246$	0.06271	0.55729	0.54845	0.0	0.0	0.03202	0.0	0.0	0.0	0.0	0.0	0.0	0.96798
Portfolio 17	$0.55729 \le f_1(x) \le 0.56246$	0.05906	0.55760	0.55041	0.0	0.0	0.03010	0.0	0.0	0.0	0.0	0.0	0.0	0.96990
Portfolio 18	$0.55760 \le f_1(x) \le 0.56246$	0.05580	0.55788	0.55214	0.0	0.0	0.02840	0.0	0.0	0.0	0.0	0.0	0.0	0.97160
Portfolio 19	$0.55788 \le f_1(x) \le 0.56246$	0.05282	0.55813	0.55373	0.0	0.0	0.02684	0.0	0.0	0.0	0.0	0.0	0.0	0.97316
Portfolio 20	$0.55813 \le f_1(x) \le 0.56246$	0.05014	0.55835	0.55516	0.0	0.0	0.02545	0.0	0.0	0.0	0.0	0.0	0.0	0.97455

The computational results presented in Table 3.7 highlight that if the investor follows aggressive strategy, a higher level of expected return is obtained, corresponding only to a higher risk level which is in sync with risk-return trade-off principle. Further, it is worthy to mention that the obtained portfolios provide the flexibility to the investor to choose the one that facilitate an effective risk-return trade-off. Fig. 3.7 shows risk-return trade-off of the obtained portfolios.

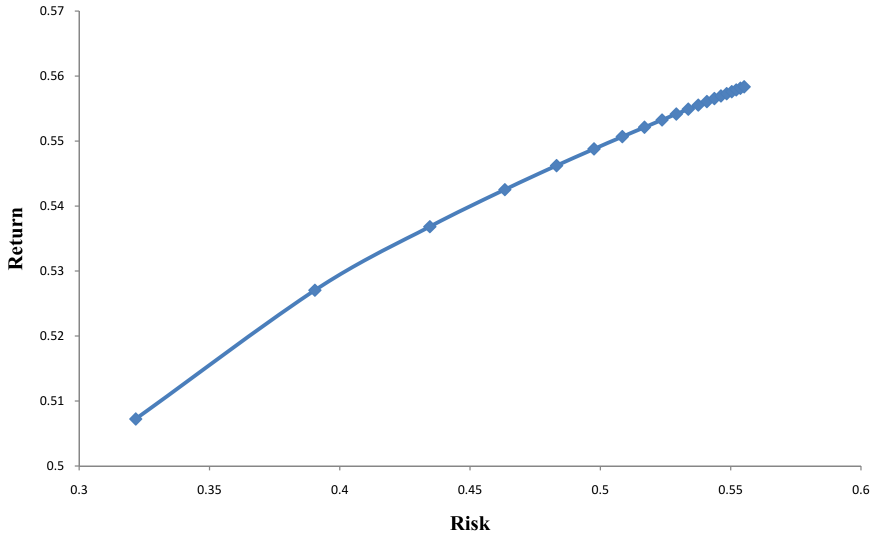

Fig. 3.7 Risk-return trade-off of the obtained portfolios presented in Table 3.7

The risk-return trade-off shown in Fig. 3.7 typically follows the diminishing-return principle, i.e., the successive increments in return pursuant to the assumption of higher level of risks get smaller and smaller. Thus, the compensation for higher level of risks becomes smaller.

• **Case 2: Conservative investor**
Suppose the investor is not satisfied with the obtained portfolio presented in Table 3.6. As the investor follows conservative strategy, the portfolio risk is improved at the cost of taking a lower return level. To do so, we consider the obtained portfolio risk value (see Table 3.5) as a new upper bound as it is lower than the worst upper bound. Further, no changes are made in the bounds of the return objective. The problem P(3.2) is solved with new parameters. Table 3.8 lists some sample preferred compromise solutions obtained by varying the upper bound of the risk objective.

Table 3.8 The proportions of the assets in the obtained portfolios by varying the upper bound of the risk objective

	Bounds	λ	Return	Risk	Allocation									
			$f_1(x)$	$f_2(x)$	ABL	ALL	BHL	CGL	HHM	HCC	KMB	MML	SIL	UNL
Portfolio 1	$0.04841 \leq f_2(x) \leq 0.20557$	0.53710	0.39305	0.12116	0.0	0.0	0.13486	0.02056	0.0	0.26699	0.0	0.21857	0.0	0.35902
Portfolio 2	$0.04841 \leq f_2(x) \leq 0.12116$	0.43814	0.35683	0.08928	0.0	0.0	0.05547	0.09079	0.0	0.25148	0.0	0.32298	0.0	0.27928
Portfolio 3	$0.04841 \leq f_2(x) \leq 0.08928$	0.36821	0.33123	0.07423	0.0	0.0	0.0	0.13961	0.00198	0.24032	0.0	0.39479	0.0	0.22331
Portfolio 4	$0.04841 \leq f_2(x) \leq 0.07423$	0.31513	0.31180	0.06609	0.0	0.0	0.0	0.12977	0.10930	0.21784	0.0	0.34513	0.0	0.19796
Portfolio 5	$0.04841 \leq f_2(x) \leq 0.06609$	0.27494	0.29710	0.06123	0.00296	0.0	0.0	0.12137	0.18861	0.20061	0.0	0.30757	0.0	0.17888
Portfolio 6	$0.04841 \leq f_2(x) \leq 0.06123$	0.24281	0.28534	0.05812	0.01776	0.0	0.0	0.11066	0.24384	0.18596	0.0	0.27775	0.0	0.16404
Portfolio 7	$0.04841 \leq f_2(x) \leq 0.05812$	0.21587	0.27548	0.05602	0.02813	0.00763	0.0	0.10125	0.28306	0.17273	0.0	0.25509	0.0	0.15210
Portfolio 8	$0.04841 \leq f_2(x) \leq 0.05602$	0.19380	0.26740	0.05455	0.03053	0.03694	0.0	0.09224	0.29387	0.15904	0.0	0.24357	0.0	0.14382
Portfolio 9	$0.04841 \leq f_2(x) \leq 0.05455$	0.17600	0.26089	0.05347	0.03245	0.06055	0.0	0.08498	0.30258	0.14800	0.0	0.23428	0.0	0.13715
Portfolio 10	$0.04841 \leq f_2(x) \leq 0.05347$	0.16121	0.25547	0.05265	0.03405	0.08018	0.0	0.07895	0.30982	0.13883	0.0	0.22656	0.0	0.13160
Portfolio 11	$0.04841 \leq f_2(x) \leq 0.05265$	0.14868	0.25089	0.05202	0.03541	0.09680	0.0	0.07385	0.31595	0.13101	0.0	0.22002	0.0	0.12690
Portfolio 12	$0.04841 \leq f_2(x) \leq 0.05202$	0.13807	0.24700	0.05152	0.03656	0.11090	0.0	0.06951	0.32115	0.12447	0.0	0.21448	0.0	0.12292
Portfolio 13	$0.04841 \leq f_2(x) \leq 0.05152$	0.12885	0.24363	0.05112	0.03756	0.12314	0.0	0.06575	0.32567	0.11875	0.0	0.20967	0.0	0.11946
Portfolio 14	$0.04841 \leq f_2(x) \leq 0.05112$	0.12084	0.24070	0.05079	0.03843	0.13376	0.0	0.06249	0.32959	0.11379	0.0	0.20549	0.0	0.11646
Portfolio 15	$0.04841 \leq f_2(x) \leq 0.05079$	0.11372	0.23809	0.05052	0.03920	0.14322	0.0	0.05958	0.33307	0.10937	0.0	0.20177	0.0	0.11379
Portfolio 16	$0.04841 \leq f_2(x) \leq 0.05052$	0.10747	0.23580	0.05029	0.03987	0.15152	0.0	0.05703	0.33613	0.10549	0.0	0.19851	0.0	0.11144

The computational results presented in Table 3.8 highlight that if the investor follows conservative strategy, a lower level of risk is obtained, corresponding to a lower return level which is in sync with risk-return trade-off principle. Fig. 3.8 shows risk-return trade-off of the obtained portfolios.

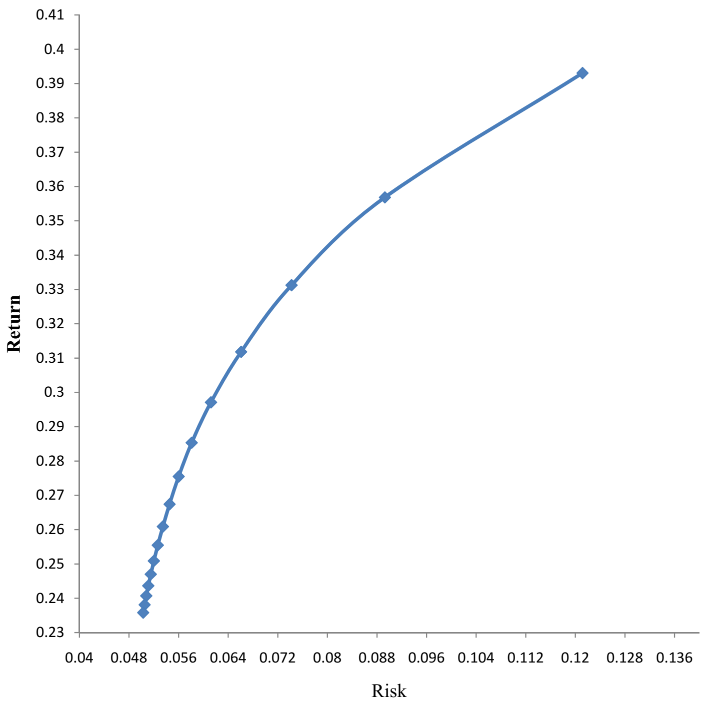

Fig. 3.8 Risk-return trade-off of the obtained portfolios presented in Table 3.8

It is clear from Fig. 3.8 that in terms of the diminishing-return principle, the fall in expected return becomes successively smaller as the conservative investor indicates successively lower levels of preferred risks.

3.5 Comments

In this chapter, we have presented the following facts:

- A bi-objective portfolio selection problem under fuzzy environment has been discussed.
- Linear membership functions have been used to represent the vague aspiration levels of investor.

- An fuzzy interactive approach has been used to obtain the solution of the fuzzy bi-objective portfolio selection model that meet the desired aspiration levels of the investor as closely as possible.
- The computational results based on real-world data have been provided to demonstrate that the solution approach is capable of generating the preferred compromise portfolio for general investor, according to the strategy being followed, i.e., aggressive or conservative strategy.
- The interactive approach has two main advantages: (i) it controls the search direction via updating lower (upper) bounds of both the objective functions; and (ii) if the investor is not satisfied with any of the obtained portfolios, several other portfolios can be generated by updating lower (upper) bounds of both the objective functions.

Chapter 4
Possibilistic Programming Approaches to Portfolio Optimization

Abstract. In this chapter, we describe possibilistic programming approaches to portfolio optimization. First we briefly introduce the foundations of possibility theory. Then we describe the portfolio selection problem with fuzzy coefficients. The problem is a fuzzy counterpart of Markowitz model. The classical possibilistic programming approaches are described. They are the fractile optimization approach, the modality optimization approach and the spread minimization approach. We show that the reduced problems become simple linear programming problems and that the solutions obtained by those approaches suggest concentrated investments or semi-concentrated investments when fuzzy coefficients are non-interactive. To obtain diversified investment solutions, regret-based possibilistic programming approach is proposed. It is shown that the reduced problem is also an linear programming problem and that the solution can be a diversified investment solution. As the other way to obtain a diversified investment solution, three models to treat the interaction among fuzzy coefficients are described. The necessity fractile optimization approach and usual minimax regret approach are applied to portfolio selection problems with interactive fuzzy coefficients. It is shown that the reduced problems are solved by linear programming techniques and that more diversified investment solutions are obtained due to the interaction among fuzzy coefficients.

4.1 Possibility Theory

Possibility theory was originally proposed by Zadeh [125] in relation to fuzzy sets. It has been developed as a theoretical foundation of fuzzy set manipulations and as a complementary theory of probability [29].

Let A and B be crisp subsets of a universal set X and u an uncertain variable which takes value in X. Under information 'u is in A', if $A \cap B \neq \emptyset$, we say that 'u is in B' is possible, and if $A \subseteq B$, we say that 'u is in B' is

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necessary. A possibility measure Π and a necessity measure N are defined as follows:

$$\Pi_A(B) = \begin{cases} 1, \text{ if } A \cap B \neq \emptyset, \\ 0, \text{ otherwise}, \end{cases} \tag{4.1}$$

$$N_A(B) = \begin{cases} 1, \text{ if } A \subseteq B, \\ 0, \text{ otherwise}. \end{cases} \tag{4.2}$$

In order to treat cases where A and B are generalized to fuzzy subsets of X, necessity and possibility measures of equations (4.1) and (4.2) are extended as follows:

Definition 4.1 (Possibility and necessity measures). Let u be an uncertain variable. The possibility and necessity measures of an event 'u is in a fuzzy subset B' under information 'u is in a fuzzy subset A' are defined as

$$\Pi_A(B) = \sup_{r \in X} \min(\mu_A(r), \mu_B(r)), \tag{4.3}$$

$$N_A(B) = \inf_{r \in X} \max(1 - \mu_A(r), \mu_B(r)), \tag{4.4}$$

respectively. To consider possibility and necessity measures, we usually assume the normality of A, i.e., $\exists\ r \in X$; $\mu_A(r) = 1$

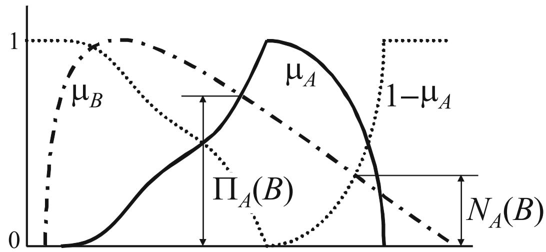


Fig. 4.1 Possibility measure and necessity measure

Possibility and necessity measures are depicted in Fig. 4.1. The possibility and necessity measures satisfy the properties shown in the following propositions and are established in [29].

Proposition 4.1. *When A is normal, i.e., $\exists\ r \in X$; $\mu_A(r) = 1$, we have*

$$N_A(B) \leq \Pi_A(B).$$

Proof. Let $\bar{r} \in X$ be a point such that $\mu_A(\bar{r}) = 1$. Then we obtain

$$\begin{aligned} N_A(B) &= \inf_{r \in X} \max(1 - \mu_A(r), \mu_B(r)) \leq \max(1 - \mu_A(\bar{r}), \mu_B(\bar{r})) \\ &= \mu_B(\bar{r}) = \min(\mu_A(\bar{r}), \mu_B(\bar{r})) \leq \sup_{r \in X} \min(\mu_A(r), \mu_B(r)) = \Pi_A(B) \end{aligned}$$

□

Proposition 4.1 shows a fact that everything which is necessary is possible. However, we do not have that $N_A(B) > 0$ implies $\Pi_A(B) = 1$.

Proposition 4.2. *We have*

$$N_A(B) = 1 - \Pi_A(B^c),$$

where B^c *is the complementary set of fuzzy set* B*. It is defined by the following membership function:*

$$\mu_{B^c}(r) = 1 - \mu_B(r), \ \forall r \in X.$$

Proof. We can prove it directly as

$$\begin{aligned} N_A(B) &= \inf_{r \in X} \max(1 - \mu_A(r), \mu_B(r)) \\ &= 1 - \sup_{r \in X} \min(\mu_A(r), 1 - \mu_B(r)) = 1 - \sup_{r \in X} \min(\mu_A(r), \mu_{B^c}(r)) = 1 - \Pi_A(B^c). \end{aligned}$$

□

Proposition 4.2 shows the duality between possibility and necessity measures, i.e., B is necessary if and only if not B is impossible.

Possibility and necessity measures can be represented by strong and weak α-level sets, where a strong α-level set $(A)_\alpha$ of fuzzy set A and a weak α-level set $[A]_\alpha$ of fuzzy set A are defined as follows for $\alpha \in \mathbf{R}$:

$$(A)_\alpha = \{r \in X \mid \mu_A(r) > \alpha\}, \quad (4.5)$$

$$[A]_\alpha = \{r \in X \mid \mu_A(r) \geq \alpha\}. \quad (4.6)$$

We have the following propositions established in [56].

Proposition 4.3. *We have*

$$\Pi_A(B) > \alpha \Leftrightarrow (A)_\alpha \cap (B)_\alpha \neq \emptyset, \quad (4.7)$$

$$N_A(B) \geq \alpha \Leftrightarrow (A)_{1-\alpha} \subseteq [B]_\alpha. \quad (4.8)$$

Proof.

$$\begin{aligned}
\Pi_A(B) > \alpha &\Leftrightarrow \sup_{r\in X} \min(\mu_A(r), \mu_B(r)) > \alpha, \\
&\Leftrightarrow \exists\, r \in X; \mu_A(r) > \alpha,\ \ \mu_B(r) > \alpha \Leftrightarrow (A)_\alpha \cap (B)_\alpha \neq \emptyset. \\
N_A(B) \geq \alpha &\Leftrightarrow \inf_{r\in X} \max(1-\mu_A(r), \mu_B(r)) \geq \alpha, \\
&\Leftrightarrow (1-\mu_A(r) < \alpha \text{ implies } \mu_B(r) \geq \alpha) \Leftrightarrow (A)_{1-\alpha} \subseteq [B]_\alpha.
\end{aligned}$$

□

From Proposition 4.3, we obtain

$$\begin{aligned}
\Pi_A(B) &= \sup\{\alpha \in \mathbf{R} \mid (A)_\alpha \cap (B)_\alpha \neq \emptyset\} \\
&= \sup\{\alpha \in \mathbf{R} \mid [A]_\alpha \cap [B]_\alpha \neq \emptyset\}, \\
N_A(B) &= \sup\{\alpha \in \mathbf{R} \mid (A)_{1-\alpha} \subseteq [B]_\alpha\} \\
&= \sup\{\alpha \in \mathbf{R} \mid [A]_{1-\alpha} \subseteq (B)_\alpha\}.
\end{aligned}$$

However, we only have

$$\begin{aligned}
\Pi_A(B) \geq \alpha &\Rightarrow [A]_\alpha \cap [B]_\alpha \neq \emptyset, \\
N_A(B) \geq \alpha &\Leftarrow [A]_{1-\alpha} \subseteq (B)_\alpha,
\end{aligned}$$

and the opposite entailments of these equations do not always satisfy.

The comparisons of closed α-level sets are more useful than those of unclosed α-level sets in programming problems with fuzzy coefficients. Then, we use the following propositions.

Proposition 4.4. *If A and B have upper semi-continuous membership functions and for any $\alpha \in (0,1]$, $[A]_\alpha$ or $[B]_\alpha$ is bounded, we have*

$$\Pi_A(B) \geq \alpha \Leftrightarrow [A]_\alpha \cap [B]_\alpha \neq \emptyset. \tag{4.9}$$

Proof. When $\Pi_A(B) = 0$, equation (4.9) is obvious because we have $[A]_\alpha = X$ and $[B]_\alpha = X$ for any $\alpha \leq 0$. Then, we assume $\Pi_A(B) = \beta > 0$. From the assumptions of the proposition, we have $[A]_{0.5\beta} \cap [B]_{0.5\beta}$ is closed and bounded, and $\min(\mu_A, \mu_B)$ is upper semi-continuous. Applying Weierstrass's theorem, we obtain

$$\begin{aligned}
\Pi_A(B) \geq \alpha &\Leftrightarrow \sup_{r\in X} \min(\mu_A(r), \mu_B(r)) \geq \alpha, \\
&\Leftrightarrow \sup_{r\in [A]_{0.5\beta} \cap [B]_{0.5\beta} \subset X} \min(\mu_A(r), \mu_B(r)) \geq \alpha, \\
&\Leftrightarrow \exists r \in X; \mu_A(r) \geq \alpha, \mu_B(r) \geq \alpha \Leftrightarrow [A]_\alpha \cap [B]_\alpha \neq \emptyset.
\end{aligned}$$

□

Proposition 4.5. *When B has an upper semi-continuous membership function, we have*

$$N_A(B) \geq \alpha \Leftrightarrow \mathrm{cl}(A)_{1-\alpha} \subseteq [B]_\alpha,$$

where $\mathrm{cl}D$ *is a closure of set* $D \subseteq X$.

Proof. It is trivial. □

Let us consider a fuzzy set A defined in $X_1 \times X_2$, which shows a possible range of variables $u_1 \in X_1$ and $u_2 \in X_2$. From the joint membership function $\mu_A : X_1 \times X_2 \to [0,1]$, we define marginal membership functions $\mu_{A[u_1]} : X_1 \to [0,1]$ and $\mu_{A[u_2]} : X_2 \to [0,1]$ by

$$\mu_{A[u_1]}(r_1) = \sup_{r_2 \in X_2} \mu_A(r_1, r_2),$$
$$\mu_{A[u_2]}(r_2) = \sup_{r_1 \in X_1} \mu_A(r_1, r_2).$$

In analogy to probability theory, we consider the following equation to define conditional membership functions $\mu_{A[u_2|u_1]} : X_2 \times X_1 \to [0,1]$ and $\mu_{A[u_1|u_2]} : X_1 \times X_2 \to [0,1]$

$$\mu_A(r_1, r_2) = \min\left(\mu_{A[u_1]}(r_1), \mu_{A[u_2|u_1]}(r_2, r_1)\right) = \min\left(\mu_{A[u_2]}(r_2), \mu_{A[u_1|u_2]}(r_1, r_2)\right). \tag{4.10}$$

Equation (4.10) was proposed by Hisdal [49] in possibility theory. Because A shows a possible range of variables $u_1 \in X_1$ and $u_2 \in X_2$, μ_A can be seen as a (joint) membership function. Accordingly, $\mu_{A[u_1]}$ and $\mu_{A[u_2]}$ can be seen as marginal membership functions and $\mu_{A[u_2|u_1]}$ and $\mu_{A[u_1|u_2]}$ can be seen as conditional membership functions.

The non-interaction and the possibilistic independence are defined as follows.

Definition 4.2 (Non-interaction and possibilistic independence). Variables u_1 and u_2 are said to be non-interactive if and only if the joint membership function μ_A and marginal membership functions $\mu_{A[u_1]}$ and $\mu_{A[u_2]}$ satisfy

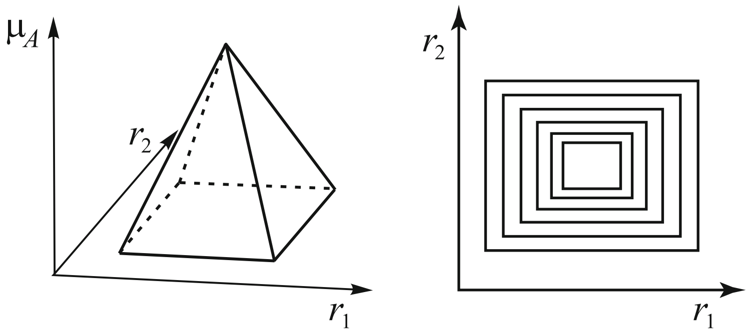
$$\mu_A(r_1, r_2) = \min\left(\mu_{A[u_1]}(r_1), \mu_{A[u_2]}(r_2)\right).$$

Variable u_1 is said to be possibilistically independent of u_2 if and only if the conditional membership function $\mu_{A[u_1|u_2]}$ equals to the marginal membership function $\mu_{A[u_1]}$ for any $r_2 \in X_2$, i.e.,

$$\mu_{A[u_1|u_2]}(r_1, r_2) = \mu_{A[u_1]}(r_1), \ \forall r_2 \in X_2.$$

Similarly, variable u_2 is said to be possibilistically independent of u_1 if and only if the conditional membership function $\mu_{A[u_2|u_1]}$ equals to the marginal membership function $\mu_{A[u_2]}$ for any $r_1 \in X_1$, i.e.,

$$\mu_{A[u_2|u_1]}(r_1, r_2) = \mu_{A[u_2]}(r_2), \ \forall r_1 \in X_1.$$



(a) 3D graph of μ_A (b) equivalence lines of μ_A

Fig. 4.2 Joint membership function μ_A of non-interactive variables u_1 and u_2 (Non-interaction)

Finally, variables u_1 and u_2 are said to be mutually independent if and only if variable u_1 is possibilistically independent of u_2 and variable u_2 is possibilistically independent of u_1.

An example of joint membership function $\mu_A : \mathbf{R}^2 \to [0,1]$ of non-interactive variables u_1 and u_2 is depicted in Fig. 4.2.

We note that u_1 and u_2 are non-interactive when u_1 is possibilistically independent of u_2. However, u_1 is not always possibilistically independent of u_2 even when u_1 and u_2 are non-interactive. Similarly, u_1 and u_2 are non-interactive when u_2 is possibilistically independent of u_1. However, u_2 is not always possibilistically independent of u_1 even when u_1 and u_2 are non-interactive. Moreover, u_2 is not always possibilistically independent of u_1 even when u_1 is possibilistically independent of u_2. Similarly, u_1 is not always possibilistically independent of u_2 even when u_2 is possibilistically independent of u_1. These can be understood by solving $\mu_A(r_1, r_2) = \min(\mu_{A[u_1]}(r_1), \mu_{A[u_2|u_1]}(r_2, r_1))$ and $\mu_A(r_1, r_2) = \min(\mu_{A[u_2]}(r_2), \mu_{A[u_1|u_2]}(r_1, r_2))$ with respect to $\mu_{A[u_2|u_1]}$ and $\mu_{A[u_1|u_2]}(r_1, r_2)$, respectively, when u_1 and u_2 are non-interactive. We obtain the following solutions:

$$\mu_{A[u_2|u_1]}(r_2, r_1) = \begin{cases} [\mu_{A[u_1]}(r_1), 1] & \text{if } \mu_{A[u_1]}(r_1) = \mu_A(r_1, r_2), \\ \mu_{A[u_2]}(r_2) & \text{if } \mu_{A[u_1]}(r_1) > \mu_A(r_1, r_2), \end{cases}$$

$$\mu_{A[u_1|u_2]}(r_1, r_2) = \begin{cases} [\mu_{A[u_2]}(r_2), 1] & \text{if } \mu_{A[u_2]}(r_2) = \mu_A(r_1, r_2), \\ \mu_{A[u_1]}(r_1) & \text{if } \mu_{A[u_2]}(r_2) > \mu_A(r_1, r_2). \end{cases}$$

Choosing any values in intervals $[\mu_{A[u_1]}(r_1), 1]$ and $[\mu_{A[u_2]}(r_2), 1]$, we obtain a solution. Namely, we have other solutions than marginal membership functions $\mu_{A[u_2]}(r_2)$ and $\mu_{A[u_1]}(r_1)$.

The concepts, joint, marginal and conditional membership functions can be extended into a fuzzy set in $\prod_{i=1}^{n} X_i = X_1 \times X_2 \times \cdots \times X_n$.

Given a function $f : \prod_{i=1}^{n} X_i \to Y$, we can extend this function from a fuzzy set in $\prod_{i=1}^{n} X_i$ to a fuzzy set in Y by the following extension principle.

Definition 4.3 (Extension principle). Given a function $f : \prod_{i=1}^{n} X_i \to Y$, the image $f(A)$ of fuzzy set $A \subseteq \prod_{i=1}^{n} X_i$ is a fuzzy set with the following membership function:

$$\mu_{f(A)}(y) = \begin{cases} \sup\limits_{\boldsymbol{r}:y=f(\boldsymbol{r})} \mu_A(\boldsymbol{r}), & \text{if } \exists\, \boldsymbol{r},\ y = f(\boldsymbol{r}), \\ 0, & \text{otherwise.} \end{cases}$$

In order to describe briefly the properties of the extension principle given by Definition 4.3, we assume $X_i = \mathbf{R}$, $i = 1, 2, \ldots, n$ and $Y = \mathbf{R}$. For strong and weak α-level sets we have the following property:

$$(f(A))_\alpha = f((A)_\alpha),\ \forall \alpha \in [0, 1), \tag{4.11}$$

$$[f(A)]_\alpha \supseteq f([A]_\alpha),\ \forall \alpha \in (0, 1], \tag{4.12}$$

where we define $f(S) = \{f(\boldsymbol{r}) \mid \boldsymbol{r} \in S\}$ for a crisp set $S \subseteq \mathbf{R}^n$.

In equations (4.11) and (4.12), the left-hand sides are strong and weak α-level sets of fuzzy set $f(A)$, the function image of fuzzy set A, while the right-hand sides are the function images of strong and weak α-level sets of A. Equation (4.11) implies that the strong α-level set of the function image of fuzzy set A equals to the function image of strong α-level set of A. On the contrary, the similar result cannot always be obtained for any weak α-level set as shown in (4.12). The only result we have is that the weak α-level set of the function image of fuzzy set A include the function image of weak α-level set of A.

However, when f is continuous, we have equality in equation (4.12) for a special fuzzy set A as shown in the following theorem [28].

Theorem 4.1. *Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function. When the weak α-level sets of A, $[A]_\alpha$, $\forall \alpha \in (0, 1]$ are closed, convex and bounded, we have*

$$[f(A)]_\alpha = f([A]_\alpha),\ \forall \alpha \in (0, 1].$$

Proof. We prove only $[f(A)]_\alpha \subseteq f([A]_\alpha)$ for an arbitrary $\alpha \in (0, 1]$, because of equation (4.12). Let $y \in [f(A)]_\alpha$. Then, $\mu_{f(A)}(y) \geq \alpha > 0$. We have

$$\mu_{f(A)}(y) = \sup_{\boldsymbol{r}:y=f(\boldsymbol{r})} \mu_A(\boldsymbol{r}) = \sup_{\boldsymbol{r} \in \{\boldsymbol{r} \mid y=f(\boldsymbol{r})\} \cap [A]_{0.5\alpha}} \mu_A(\boldsymbol{r}) \geq \alpha. \tag{4.13}$$

Because f is continuous and $[A]_{0.5\alpha}$ is closed and bounded, the set $\{\boldsymbol{r} \mid y = f(\boldsymbol{r})\} \cap [A]_{0.5\alpha}$ is closed and bounded. Moreover, $[A]_\alpha$, $\forall \alpha \in (0, 1]$ are closed and bounded if and only if μ_A is upper semi-continuous. Applying Weierstrass's theorem, from (equation 4.13), we obtain

$$\exists\, \boldsymbol{r};\ y = f(\boldsymbol{r}) \text{ and } \boldsymbol{r} \in [A]_\alpha$$

This implies $y \in f([A]_\alpha)$.

Hence, we obtain $[f(A)]_\alpha \subseteq f([A]_\alpha)$. □

Let us consider a case where the membership function μ_A of $A \subseteq \mathbf{R}^n$ is the joint membership function of n non-interactive variables u_i, $i = 1, 2, \ldots, n$. Let μ_{A_i} be the marginal membership function on u_i and A_i be fuzzy set defined by μ_{A_i} $(i = 1, 2, \ldots, n)$. We may apply the following corollary:

Corollary 4.1. *Consider a fuzzy set $A \subseteq \mathbf{R}^n$ defined by membership function*

$$\mu_A(r_1, r_2, \ldots, r_n) = \min\left(\mu_{A_1}(r_1), \mu_{A_2}(r_2), \ldots, \mu_{A_n}(r_n)\right),$$

where μ_{A_i} is the membership function of $A_i \subseteq R$ $(i = 1, 2, \ldots, n)$. When the weak α-level sets $[A_i]_\alpha$, $\alpha \in (0,1]$ of A_i are closed and bounded intervals for $i = 1, 2, \ldots, n$, we have

$$[f(A)]_\alpha = f\left([A_1]_\alpha, [A_2]_\alpha, \ldots, [A_n]_\alpha\right), \ \forall \alpha \in (0,1].$$

Especially, when f is a weighted sum, i.e., $f(r_1, r_2, \ldots, r_n) = \sum_{i=1}^n w_i r_i$ and when $[A_i]_\alpha$, $\alpha \in (0,1]$ are closed and bounded intervals $[a_i^{\mathrm{L}}(\alpha), a_j^{\mathrm{R}}(\alpha)]$ for $i = 1, 2, \ldots, n$, we obtain

$$[f(A)]_\alpha = [f^{\mathrm{L}}(\alpha), f^{\mathrm{R}}(\alpha)], \ \forall \alpha \in [0,1],$$

where $f^{\mathrm{L}} : (0,1] \to \mathbf{R}$ and $f^{\mathrm{R}} : (0,1] \to \mathbf{R}$ are defined by

$$f^{\mathrm{L}}(\alpha) = \sum_{i:w_i \geq 0} w_i a_i^{\mathrm{L}}(\alpha) + \sum_{i:w_i < 0} w_i a_i^{\mathrm{R}}(\alpha),$$

$$f^{\mathrm{R}}(\alpha) = \sum_{i:w_i \geq 0} w_i a_i^{\mathrm{R}}(\alpha) + \sum_{i:w_i < 0} w_i a_i^{\mathrm{L}}(\alpha).$$

We define fuzzy numbers as follows.

Definition 4.4 (Fuzzy number). A fuzzy set A of real line $\mathbf{R}$ is said to be a fuzzy number if and only if A satisfies the following four requirements:

(i) A is normal, i.e., $\exists\, r \in \mathbf{R}$; $\mu_A(r) = 1$, or equivalently, $[A]_1 \neq \emptyset$

(ii) A is convex, i.e., μ_A is a quasi-concave function ($\forall \kappa \in [0,1]$, $\forall r_1, r_2 \in \mathbf{R}$; $\mu_A(\kappa r_1 + (1-\kappa) r_2) \geq \min(\mu_A(r_1), \mu_A(r_2))$, or equivalently, $[A]_\alpha$, $\forall \alpha \in (0,1]$ is convex

(iii) A has an upper semi-continuous membership function μ_A, or equivalently, $[A]_\alpha$, $\forall \alpha \in (0,1]$ is closed

(iv) A is bounded, i.e., $\lim_{r \to +\infty} \mu_A(r) = \lim_{r \to -\infty} \mu_A(r) = 0$, or equivalently, $[A]_\alpha$, $\forall \alpha \in (0,1]$ is bounded

For a continuous function values of fuzzy numbers, we can apply Theorem 4.1.

4.2 Portfolio Selection Using Non-interactive Coefficients

The Markowitz model [91] is famous for portfolio selection. In this model, an expected return rate of a bond is treated as a random variable. Stochastic programming is applied so that the solution is obtained by minimizing the variance of the total expected return rate under the constraint that the mean of the total expected return rate is equal to a predetermined value. This model yields a diversified investment solution unless the expected return rates are completely positively dependent on each other. In the traditional portfolio theory, a diversified investment has been often regarded as a good policy to reduce the risk.

Possibilistic programming is a known similar approach to stochastic programming; therefore, an application of possibilistic programming to portfolio selection is conceivable. In possibilistic programming approaches, the expected return rates are not treated as random variables but as variables whose possible ranges are given by fuzzy numbers. Application of possibilistic programming to portfolio selection may have a two-fold advantage [59]:

(i) The knowledge of the expert can be easily used for the estimation of the return rates
(ii) The reduced problem is more tractable than that of the stochastic programming approach

However, because classical possibilistic programming approaches have been developed under the implicit assumption that all uncertain variables are non-interactive one and their applications to the portfolio selection do not yield diversified investments. In Section 4.2.2, we observe the tractability of the reduced problems of possibilistic programming approaches and also that their solutions are not diversified investments. In Section 4.2.3 considering how an optimization model yields a diversified investment solution, a novel possibilistic programming approach to the portfolio selection is shown. This approach is based on regret which the decision maker may undertake. More concretely, a minimax regret approach to the possibilistic portfolio selection is discussed. It is shown that a diversified investment solution is obtained by this approach even if uncertain variables are non-interactive. In Section 4.2.4 some examples are given in order to compare the solutions obtained by the classical and novel approaches.

4.2.1 Possibilistic Portfolio Selection Problem

When rate of return of all assets are known exactly, a portfolio selection problem can be formulated as

$$\begin{aligned}\mathbf{P(4.1)}\quad & \max\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\\ & \text{subject to}\\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}=1\,,\\ & \boldsymbol{x}\geq 0\,,\end{aligned}$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_n)^{\mathrm{T}}$, $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$ and $\boldsymbol{e} = (1, 1, \ldots, 1)^{\mathrm{T}}$. The component c_i represents the rate of return of the i-th asset and x_i represent the proportion of the total funds invested in the i-th asset. Thus, the problem is maximizing the total return rate. An optimal solution can be obtained easily as shown in the following theorem.

Theorem 4.2. *An optimal solution to problem P(4.1) is a concentrated investment on an asset which have the maximum rate of return. Namely, a solution* $x_{i^*} = 1$, $x_i = 0$, $\forall i \neq i^*$ *where* $c_{i^*} \geq c_i$, $i = 1, 2, \ldots, n$ *is an optimal solution to problem P(4.1).*

Proof. Trivial. □

However, in the real setting, one can seldom obtain the return rate without any uncertainty. Thus, the decision maker should make decisions under uncertainty. Such an uncertainty has been treated as a random variable so far. The problem has been formulated as the following stochastic programming problem:

$$\begin{aligned}\mathbf{P(4.2)}\quad & \max\ \boldsymbol{C}^{\mathrm{T}}\boldsymbol{x}\\ & \text{subject to}\\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}=1\,,\\ & \boldsymbol{x}\geq 0\,,\end{aligned}$$

where $\boldsymbol{C} = (C_1, C_2, \ldots, C_n)^{\mathrm{T}}$ is a random variable vector obeying a multivariate probability distribution which have a mean vector $\boldsymbol{m}$ and a covariance matrix V. To such a problem, Markowitz [91] proposed the following model to determine the investment rates $\boldsymbol{x}$:

$$\begin{aligned}\mathbf{P(4.3)}\quad & \min\ \boldsymbol{x}^{\mathrm{T}}V\boldsymbol{x}\\ & \text{subject to}\\ & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\geq z_0\,,\\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}=1\,,\\ & \boldsymbol{x}\geq 0\,,\end{aligned}$$

where z_0 is a predetermined desirable expected return rate. In problem P(4.3), equality constraint $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = z_0$ of the original Markowitz model is replaced with inequality constraint $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z_0$. This replacement enables the model to generate a suitable solution even for an underestimated expected return rate z_0.

To formulate problem P(4.3), we need to estimate the probability distribution, strictly speaking, a mean vector $\boldsymbol{c}$ and a covariance matrix V. This is not an easy task when the number of assets is large. Therefore, several simplified methods, sacrificing generality, have been proposed. Moreover, it is difficult to reflect the unquantifiable factors such as the knowledge of experts, the trend of public opinion, and so on. Furthermore, even though the probability distribution can be estimated, there is no guarantee that the return rates truly obey it.

The variation range of an uncertain return rate can be represented by a fuzzy number other than a probability distribution. In this approach, although the uncertain values are estimated substantially, the unquantifiable factors such as the knowledge of experts can be reflected easily. Then, it may be worthwhile to introduce fuzzy numbers to the portfolio selection problem.

Treating each return rate as a variable γ_i whose variation range is represented by a fuzzy number C_i, we have the following portfolio selection problem with fuzzy numbers:

$$\begin{aligned} \textbf{P(4.4)} \quad & \max \ \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x} \\ & \text{subject to} \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,, \end{aligned}$$

where $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_n)^{\mathrm{T}}$ is a variable vector whose variation range is represented by an n-dimensional fuzzy set C. An easiest way to obtain n-dimensional fuzzy set C is to define it with non-interaction assumption by

$$\mu_C(\boldsymbol{c}) = \min_{i=1,2,\ldots,n} \mu_{C_i}(c_i), \tag{4.14}$$

where μ_C is the membership function of C and μ_{C_i} is the membership function of the variation range C_i of the return rate γ_i of the i-th asset ($i = 1, 2, \ldots, n$). Namely, we assume that γ_i, $i = 1, 2, \ldots, n$ are non-interactive one another. In this section, we consider C defined by the membership function of equation (4.14). For the sake of simplicity, we assume that μ_C is upper semi-continuous, in other words, each μ_{C_i} is upper semi-continuous. Moreover, we assume

$$\lim_{c_i \to -\infty} \mu_{C_i}(c_i) = \lim_{c_i \to +\infty} \mu_{C_i}(c_i) = 0, \ \ i = 1, 2, \ldots, n.$$

By these assumptions, $[C_i]_\alpha$, $\alpha \in (0, 1]$ become closed and bounded intervals.

4.2.2 The Classical Possibilistic Programming Approaches

Various approaches have been proposed to a possibilistic programming problem. In this section, we apply major possibilistic programming approaches

[57, 58, 59] to the portfolio selection problem P(4.4) with fuzzy numbers. We assume that the decision maker's attitude is uncertainty (risk) averse [56]. From this point of view, proper possibilistic approaches are introduced.

- **Fractile optimization approach**

Given an appropriate level $\alpha^0 \in (0, 1]$, problem P(4.4) is formulated so as to maximize a return rate z under a constraint that a necessity measure of the event that the objective function value is not less than z is greater than or equal to α^0:

$$\begin{aligned} \mathbf{P(4.5)}\quad & \max\ z \\ & \text{subject to} \\ & N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z\}) \geq \alpha^0\,, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,, \end{aligned}$$

where N_C is a necessity measure under a possible range C of rate of return of assets. From the definition of the necessity measure N_C, $N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z\})$ shows a necessity degree to what extent the objective function value is not less than z. It is represented as

$$\begin{aligned} N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z\}) &= \inf_{\substack{\boldsymbol{c} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} < z}} (1 - \mu_C(\boldsymbol{c})) \\ &= \inf_{y<z} \left(1 - \sup_{\substack{\boldsymbol{c} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = y}} \mu_C(\boldsymbol{c}) \right) \\ &= \inf_{y<z} (1 - \mu_{C^{\mathrm{T}}\boldsymbol{x}}(y)) = N_{C^{\mathrm{T}}\boldsymbol{x}}([z, +\infty)), \end{aligned} \tag{4.15}$$

where we define fuzzy set $C^{\mathrm{T}}\boldsymbol{x}$ by the extension principle, i.e., the membership function of $C^{\mathrm{T}}\boldsymbol{x}$ is defined by

$$\mu_{C^{\mathrm{T}}\boldsymbol{x}}(y) = \sup_{\substack{\boldsymbol{c} \\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = y}} \mu_C(\boldsymbol{c}).$$

Problem P(4.5) is called a necessity fractile optimization model because z corresponds to a fractile of probability distribution where the probability measure is replaced with a necessity measure.

Applying Proposition 4.5, we obtain

$$N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z\}) \geq \alpha^0 \Leftrightarrow N_{C^{\mathrm{T}}\boldsymbol{x}}((-\infty, z]) \geq \alpha^0 \Leftrightarrow \mathrm{cl}(C^{\mathrm{T}}\boldsymbol{x})_{1-\alpha^0} \subseteq (-\infty, z]. \tag{4.16}$$

Under the non-interaction equation (4.14) and the non-negativity of $\boldsymbol{x}$, similar to Corollary 4.1, we have

$$\mathrm{cl}(C^{\mathrm{T}}\boldsymbol{x})_{1-\alpha^0} = \left[\boldsymbol{c}^{\mathrm{L}}(1-\alpha^0)^{\mathrm{T}}\boldsymbol{x}, \boldsymbol{c}^{\mathrm{R}}(1-\alpha^0)^{\mathrm{T}}\boldsymbol{x}\right], \tag{4.17}$$

where $\boldsymbol{c}^{\mathrm{L}}(\cdot) = (c_1^{\mathrm{L}}(\cdot), c_2^{\mathrm{L}}(\cdot), \ldots c_n^{\mathrm{L}}(\cdot))^{\mathrm{T}}$, $\boldsymbol{c}^{\mathrm{R}}(\cdot) = (c_1^{\mathrm{R}}(\cdot), c_2^{\mathrm{R}}(\cdot), \ldots c_n^{\mathrm{R}}(\cdot))^{\mathrm{T}}$ and

$$c_i^{\mathrm{L}}(\alpha) = \inf\{q \mid \mu_{C_i}(q) > \alpha\},$$
$$c_i^{\mathrm{R}}(\alpha) = \sup\{q \mid \mu_{C_i}(q) > \alpha\}.$$

Then, from equation (4.16), problem P(4.5) is reduced to the following liner programming problem [58]:

$$\begin{aligned} \mathbf{P(4.6)}\ \ & \max\ \boldsymbol{c}^{\mathrm{L}}(1-\alpha^0)^{\mathrm{T}}\boldsymbol{x} \\ & \text{subject to} \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{aligned}$$

• Modality optimization approach

Given a target value z^0, problem P(4.4) is formulated so as to maximize a necessity measure of the event that the objective function value is not less than z^0:

$$\begin{aligned} \mathbf{P(4.7)}\ \ & \max\ N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z^0\}) \\ & \text{subject to} \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{aligned}$$

This problem is called a necessity measure optimization model.

Because problem P(4.7) is equivalent to

$$\begin{aligned} \mathbf{P(4.8)}\ \ & \max\ \alpha \\ & \text{subject to} \\ & N_C(\{\boldsymbol{c} \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z^0\}) \geq \alpha\,, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{aligned}$$

From equations (4.16) and (4.17), the above problem reduces to the following non-linear programming problem [58]:

$$\begin{array}{ll}\mathbf{P(4.9)} & \max\ \alpha \\ & \text{subject to} \\ & \boldsymbol{c}^{\mathrm{L}}(1-\alpha)^{\mathrm{T}}\boldsymbol{x} \geq z^0\,, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{array}$$

When $\mu_{C_i}(c_i)$, $i = 1, 2, \ldots, n$ are expressed by

$$\mu_{C_i}(c_i) = \begin{cases} L\left(\dfrac{\bar{c}_i^{\mathrm{L}} - c_i}{s_i^{\mathrm{L}}}\right) & \text{if } c_i < \bar{c}_i^{\mathrm{L}}, \\ 1 & \text{if } \bar{c}_i^{\mathrm{L}} \leq c_i \leq \bar{c}_i^{\mathrm{R}}, \\ R_i\left(\dfrac{c_i - \bar{c}_i^{\mathrm{R}}}{s_i^{\mathrm{R}}}\right) & \text{if } c_i > \bar{c}_i^{\mathrm{R}}, \end{cases}$$

where $\bar{c}_i^{\mathrm{L}}$, $\bar{c}_i^{\mathrm{R}}$, $s_i^{\mathrm{L}} > 0$ and $s_i^{\mathrm{R}} > 0$ are constants and $L : [0, +\infty) \to [0, 1]$ and $R_i : [0, +\infty) \to [0, 1]$ are reference functions such that (i) $L(0) = R_i(0) = 1$, (ii) L and R_i are upper semi-continuous and non-increasing functions, and (iii) $\lim_{r\to+\infty} L(r) = \lim_{r\to+\infty} R_i(r) = 0$. We note that L are common for all C_i, $i = 1, 2, \ldots, n$ while R_i may depend on C_i.

For $\alpha \in [0, 1]$ we define

$$L^*(\alpha) = \begin{cases} \sup\{r \in [0, +\infty) \mid L(r) > \alpha\}, & \text{if } \alpha < 1, \\ 0, & \text{if } \alpha = 1. \end{cases} \tag{4.18}$$

We obtain

$$c_i^{\mathrm{L}}(\alpha) = \bar{c}_i^{\mathrm{L}} - L^*(\alpha)s_i^{\mathrm{L}}.$$

Then because $L^*(\cdot)$ is non-increasing, problem P(4.9) is reduced to

$$\begin{array}{ll}\mathbf{P(4.10)} & \max\ L^*(1-\alpha) \\ & \text{subject to} \\ & (\bar{\boldsymbol{c}}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x} - L^*(1-\alpha)(\boldsymbol{s}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x} \geq z^0\,, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{array}$$

Finally, the above problem reduces to the following linear fractional programming problem:

$$\begin{array}{ll}\mathbf{P(4.11)} & \max\ \dfrac{(\bar{\boldsymbol{c}}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x} - z^0}{(\boldsymbol{s}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{x}} \\ & \text{subject to} \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{array}$$

As is known in the literature, the solution to problem P(4.11) can be obtained by solving the following linear programming problem:

$$
\begin{aligned}
\mathbf{P(4.12)}\quad & \max\ (\bar{\boldsymbol{c}}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{y} - z^0 t \\
& \text{subject to} \\
& (\boldsymbol{s}^{\mathrm{L}})^{\mathrm{T}}\boldsymbol{y} = 1\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{y} = t\,, \\
& \boldsymbol{y} \geq 0\,, \\
& t \geq 0\,.
\end{aligned}
$$

Let $(\hat{\boldsymbol{y}}, \hat{t})$ be an optimal solution of problem P(4.12), then the solution to problem P(4.11) is obtained as $\hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}/\hat{t}$.

• **Spread minimization approach**
Define a representative vector $\hat{\boldsymbol{c}}$ of fuzzy set $C \subseteq \mathbf{R}^n$. Given $\alpha^0 \in (0, 1]$ and z^0, the width of the α^0-level set $[C^{\mathrm{T}}\boldsymbol{x}]_{\alpha^0} = \{y \mid \mu_{C^{\mathrm{T}}\boldsymbol{x}}(y) \geq \alpha^0\}$ of fuzzy set $C^{\mathrm{T}}\boldsymbol{x}$ can be minimized under the constraint $\hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x} \geq z^0$. Problem P(4.4) is formulated as

$$
\begin{aligned}
\mathbf{P(4.13)}\quad & \min\ w \\
& \text{subject to} \\
& \max_{y^{\mathrm{R}}, y^{\mathrm{L}} \in [C\boldsymbol{x}^{\mathrm{t}}]_{\alpha^0}} (y^{\mathrm{R}} - y^{\mathrm{L}}) \leq w\,, \\
& \hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x} \geq z^0\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& t\boldsymbol{x} \geq 0\,.
\end{aligned}
$$

and reduced to the following linear programming problem [58]:

$$
\begin{aligned}
\mathbf{P(4.14)}\quad & \min\ (\boldsymbol{c}_{\mathrm{R}}(\alpha^0)^{\mathrm{T}} - \boldsymbol{c}_{\mathrm{L}}(\alpha^0)^{\mathrm{T}})\boldsymbol{x} \\
& \text{subject to} \\
& \hat{\boldsymbol{c}}\boldsymbol{x}^{\mathrm{t}} \geq z^0\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \geq 0\,,
\end{aligned}
$$

where $\boldsymbol{c}_{\mathrm{L}}(\cdot) = (c_{1\mathrm{L}}(\cdot), c_{2\mathrm{L}}(\cdot), \ldots, c_{n\mathrm{L}}(\cdot))^{\mathrm{T}}$, $\boldsymbol{c}_{\mathrm{R}}(\cdot) = (c_{1\mathrm{R}}(\cdot), c_{2\mathrm{R}}(\cdot), \ldots, c_{n\mathrm{R}}(\cdot))^{\mathrm{T}}$ and

$$
\begin{aligned}
c_{i\mathrm{L}}(\alpha) &= \inf\{q \mid \mu_{C_i}(q) \geq \alpha\}, \\
c_{i\mathrm{R}}(\alpha) &= \sup\{q \mid \mu_{C_i}(q) \geq \alpha\}.
\end{aligned}
$$

Since the variance of a probability distribution corresponds to the spread of a fuzzy set, this model can be seen as a counterpart of the Markowitz model.

• **Classical fuzzy programming approach**

We have the following theorem.

Theorem 4.3. *A concentrated investment solution such that* $x_i = 1$ *for some* $i \in \{1, 2, \ldots, n\}$ *is an optimal solution to problem P(4.6). The same assertion is valid for problem P(4.9). A semi-concentrated investment solution such that* $x_i + x_j = 1$ *for some* $i,\ j \in \{1, 2, \ldots, n\}$ *is an optimal solution to problem P(4.14).*

Proof. The first and third assertion of this theorem are obvious when taking into account the number of constraints excluding non-negativity constraints of the problem. Indeed, problem P(4.6) is a linear programming problem with one constraint and problem P(4.14) is a linear programming problem with two constraints. Since a linear programming problem can be solved by the simplex method, problems P(4.6) and P(4.14) have one and two basic variables, respectively. Hence concentrated and semi-concentrated investment solutions are optimal to problems P(4.6) and P(4.14), respectively. Now, we prove the second assertion. Let $(\hat{\boldsymbol{x}}, \hat{\alpha})$ be an optimal solution to problem P(4.9). Consider an optimal solution $\boldsymbol{x}^*$ to a linear programming problem,

$$\begin{aligned} \textbf{P(4.15)}\quad & \max\ \boldsymbol{c}^{\mathrm{L}}(1-\hat{\alpha})^{\mathrm{T}}\boldsymbol{x} \\ & \text{subject to} \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ & \boldsymbol{x} \geq 0\,. \end{aligned}$$

The solution $(\boldsymbol{x}^*, \hat{\alpha})$ is also an optimal solution to problem P(4.9). By the same way of the proof of the first and second assertions, an optimal solution to problem P(4.15) is a concentrated investment solution. □

Theorem 4.3 shows that if each of problems P(4.6), P(4.9) and P(4.14) has a unique solution, no diversified investment solution is obtained. Moreover, when those problems are solved by the simplex method, the conventional possibilistic programming approaches to the portfolio selection do not yield a solution compatible with the traditional portfolio theory.

4.2.3 Regret-Based Possibilistic Programming Approach

Inuiguchi and Tanino [63] discussed why a diversified investment solution under independent return rate assumption is preferred by a decision maker who has an uncertainty (risk) averse attitude. They pointed out the following two reasons:

(a) **Property of a measure.** Consider the event that the total return rate is not less than a certain value. When the measure of the event under a diversified investment solution is greater than that of the event under a concentrated investment solution, the diversified investment solution should be preferred. In the other words, the uncertainty is decreased by distribution to many assets.

(b) **The worst regret criterion.** Suppose that the investor has invested his money in an asset according to a concentrated investment solution. If the rate of return of another asset becomes better than that of the invested asset as a result, the investor may feel a regret. At the decision making stage, we cannot know the return rate determined in the future. Thus, any concentrated investment solution may bring regret to the decision maker. In this sense, if the decision maker is interested in minimizing the worst regret which may be undertaken, a diversified investment solution must be preferred.

Markowitz model [91] and the other stochastic programming approaches yield a diversified investment solution because of (a). Indeed, we have

$$\mathrm{Prob}(\lambda X_1 + (1-\lambda)X_2 \geq k) > \mathrm{Prob}(X_i \geq k),\ \forall\lambda \in (0,1),\ i = 1,2,$$

when independent random variables X_1 and X_2 obey the same marginal normal (probability) distribution. Moreover we have

$$\mathrm{Var}(\lambda X_1 + (1-\lambda)X_2) < \mathrm{Var}(X_i),\ \forall\lambda \in (0,1),\ i = 1,2,$$

where $\mathrm{Var}(X)$ is the variance, i.e., an uncertainty criterion, of a random variable X.

In possibilistic programming approaches, we could not obtain a diversified investment solution since possibility and necessity measures do not have the property mentioned in (a). For possibility and necessity measures, we have

$$\mathrm{Pos}(\lambda X_1 + (1-\lambda)X_2 \geq k) = \mathrm{Pos}(X_i \geq k),\ \forall\lambda \in [0,1],\ i = 1,2,$$

$$\mathrm{Nes}(\lambda X_1 + (1-\lambda)X_2 \geq k) = \mathrm{Nes}(X_i \geq k),\ \forall\lambda \in [0,1],\ i = 1,2,$$

where X_1 and X_2 are non-interactive uncertain variables whose possible ranges are expressed by the same marginal fuzzy sets. Moreover, in the same setting of X_1 and X_2, we have

$$\mathrm{Spd}(\lambda X_1 + (1-\lambda)X_2) = \mathrm{Spd}(X_i),\ \forall\lambda \in [0,1],\ i = 1,2,$$

where $\mathrm{Spd}(X)$ is the spread of fuzzy sets representing uncertain variable X, i.e.,

$$\mathrm{Spd}(X) = \sup\{r \mid \mu_X(r) > 0\} - \inf\{r \mid \mu_X(r) > 0\}.$$

Hence, the conventional possibilistic programming approaches fail to yield a diversified investment solution without introducing the concept of the regret or the interaction among uncertain variables. In what follows, we describe an approach proposed by Inuiguchi and Tanino [63]. They introduced regret into the possibilistic portfolio selection problem P(4.4).

Suppose that a decision maker is informed about the determined return rates $\boldsymbol{c}$ after he/she has invested his/her money in assets according to a feasible solution $\boldsymbol{x}$ to problem P(4.4), he/she will have a regret $r(\boldsymbol{x};\boldsymbol{c})$ which can be quantified as

$$r(\boldsymbol{x};\boldsymbol{c}) = \max_{\substack{\boldsymbol{y} \\ \boldsymbol{e}^{\mathrm{T}}\boldsymbol{y}=1,\ \boldsymbol{y}\geq 0}} F(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{y},\ \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}), \tag{4.19}$$

where $F : D_1 \times D_2 \to \mathbf{R}$ $(D_1, D_2 \subseteq \mathbf{R})$ is a continuous function such that $F(\cdot;r)$ is strictly increasing and $F(r;\cdot)$ is strictly decreasing. These properties of the function F reflect the fact that the right side of equation (4.19) evaluates the regret $r(\boldsymbol{x};\boldsymbol{c})$. Since F is continuous, so is $r(\cdot;\cdot)$.

At the decision making stage, the decision maker cannot know the return rate $\boldsymbol{c}$ determined in the future but a possible range C with membership function μ_C. By the extension principle (Definition 4.3), the possible range of regret $r(\boldsymbol{x})$ can be obtained by a fuzzy set $R(\boldsymbol{x})$ having the following membership function:

$$\mu_{R(\boldsymbol{x})}(r) = \sup_{\substack{\boldsymbol{c} \\ r=r(\boldsymbol{x};\boldsymbol{c})}} \mu_C(\boldsymbol{c}). \tag{4.20}$$

We regard the possibilistic portfolio selection problem P(4.1) as a problem of minimizing a regret $r(\boldsymbol{x})$, i.e.,

$$\begin{aligned}
\textbf{P(4.16)}\quad & \min\ r(\boldsymbol{x}) \\
& \text{subject to} \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \geq 0\,.
\end{aligned}$$

Since $r(\boldsymbol{x})$ is an uncertain value whose possible range is given by a fuzzy set $R(\boldsymbol{x})$, problem P(4.16) is a programming problem with fuzzy parameters. Thus, we can apply a possibilistic programming approach. We apply the fractile model to problem P(4.16) so that, given α^0, problem P(4.16) is formulated as

$$\begin{aligned}
\textbf{P(4.17)}\quad & \min\ z \\
& \text{subject to} \\
& N_{R(\boldsymbol{x})}(\{r \mid r \leq z\}) \geq \alpha^0\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \geq 0\,.
\end{aligned}$$

Now we transform problem P(4.17) to a linear programming problem. Using equations (4.4) and (4.20), we have

$$N_{R(x)}(\{r \mid r \le z\}) = \inf_{r>z}\left(1 - \mu_{R(\boldsymbol{x})}(r)\right) = \inf_{r>z}\left(1 - \sup_{\substack{\boldsymbol{c} \\ r=r(\boldsymbol{x};\boldsymbol{c})}} \mu_C(\boldsymbol{c})\right)$$

$$= \inf_{\substack{\boldsymbol{c} \\ r(\boldsymbol{x};\boldsymbol{c})>z}} (1 - \mu_C(\boldsymbol{c})).$$

Thus, $N_{R(x)}(\{r \mid r \le z\}) \ge \alpha^0$ is equivalent to

$$\mu_C(\boldsymbol{c}) > 1 - \alpha^0 \ \text{ implies } \ r(\boldsymbol{x};\boldsymbol{c}) \le z. \tag{4.21}$$

By the continuity of $r(\boldsymbol{x};\boldsymbol{c})$ with respect to $\boldsymbol{c}$, equation (4.21) becomes equivalent to

$$\sup_{\boldsymbol{c}\in(C)_{1-\alpha^0}} r(\boldsymbol{x};\boldsymbol{c}) \le z, \tag{4.22}$$

where $(C)_{1-\alpha^0}$ is a $(1-\alpha^0)$–level set, i.e., $(C)_{1-\alpha^0} = \{\boldsymbol{c} \mid \mu_C(\boldsymbol{c}) > 1-\alpha^0\}$. A closure of the $(1-\alpha^0)$–level set, cl $(C)_{1-\alpha^0}$, can be expressed as

$$\text{cl } (C)_{1-\alpha^0} = \{\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \mid c_i^{\text{L}}(1-\alpha^0) \le c_i \le c_i^{\text{R}}(1-\alpha^0),\ i = 1, 2, \ldots, n\},$$

where $c_i^{\text{L}}(\cdot)$ is a function defined by (4.18) and $c_i^{\text{R}}(\cdot)$ is defined by

$$c_i^{\text{R}}(\alpha) = \sup\{q \mid \mu_{C_i}(q) > \alpha\}.$$

By the continuity of $r(\boldsymbol{x};\boldsymbol{c})$ with respect to $\boldsymbol{c}$, the supremum, 'sup', and the $(1-\alpha^0)$–level set, $(C)_{1-\alpha^0}$, can be replaced with the maximum, 'max', and the closure, cl $(C)_{1-\alpha^0}$ in (4.22), respectively. Hence, we have

$$N_{R(x)}(\{r \mid r \le z\}) \ge \alpha^0 \quad \Leftrightarrow \quad \max_{\boldsymbol{c}\in\text{cl } (C)_{1-\alpha^0}} r(\boldsymbol{x};\boldsymbol{c}) \le z. \tag{4.23}$$

Using equations (4.19) and (4.23) into problem P(4.17), we have

$$\begin{aligned}
\textbf{P(4.18)}\quad & \min\ z \\
& \text{subject to} \\
& \max_{\substack{\boldsymbol{c},\boldsymbol{y} \\ \boldsymbol{c}\in\text{cl } (C)_{1-\alpha^0} \\ \boldsymbol{e}^{\text{T}}\boldsymbol{y}=1,\ \boldsymbol{y}\ge 0}} F(\boldsymbol{c}^{\text{T}}\boldsymbol{y}, \boldsymbol{c}^{\text{T}}\boldsymbol{x}) \le z\,, \\
& \boldsymbol{e}^{\text{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \ge 0\,.
\end{aligned}$$

Since cl $(C)_{1-\alpha^0} = [c_i^{\text{L}}(1-\alpha^0), c_i^{\text{R}}(1-\alpha^0)]$, problem P(4.18) is a minimax problem with linear constraints. From the assumption of F and Theorem 4.2, we have

$$\max_{\substack{c,y \\ c\in \mathrm{cl}\,(C)_{1-\alpha^0} \\ e^{\mathrm{T}}y=1,\ y\geq 0}} F(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}) = \max_{c\in \mathrm{cl}\,(C)_{1-\alpha^0}} F\left(\max_{e^{\mathrm{T}}y=1,\ y\geq 0} \boldsymbol{c}^{\mathrm{T}}\boldsymbol{y}, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}\right)$$

$$= \max_{i=1,2,\ldots,n} \max_{c\in \mathrm{cl}\,(C)_{1-\alpha^0}} F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}),$$

where $\boldsymbol{e}_i$ is a unit vector whose i-th component is one. Hence, problem P(4.18) is reduced to

P(4.19) min z

subject to

$$\max_{c\in \mathrm{cl}\,(C)_{1-\alpha^0}} F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}) \leq z,\ i = 1, 2, \ldots, n\,,$$

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,,$$

$$\boldsymbol{x} \geq 0\,.$$

Problem P(4.19) can be solved by a relaxation procedure and non-linear programming techniques. Moreover, we consider the case when F can be expressed as

$$F(r_1, r_2) = \varphi(f(r_1)r_2 + g(r_1)), \tag{4.24}$$

where $\varphi : \mathbf{R} \to \mathbf{R}$ is strictly increasing, $f : D_1 \to \mathbf{R}$ and $g : D_2 \to \mathbf{R}$ satisfies

(i) $f(r) < 0,\ \forall r \in [L, R]$
(ii) $f'(r_1)r_2 + g'(r_1) > 0,\ \forall (r_1, r_2) \in [L, R] \times [L, R]$
(iii) For all $i \in \{1, 2, \ldots, n\}$

$$\inf_{\substack{c,x \\ c\in (C)_0 \\ e^{\mathrm{T}}x=1,\ x\geq 0}} (f'(c_i)\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + g'(c_i) + f(c_i)x_i) \geq 0.$$

L and R are defined by

$$L = \min_{i=1,2,\ldots,n} c_i^{\mathrm{L}}(0),\ R = \max_{i=1,2,\ldots,n} c_i^{\mathrm{R}}(0).$$

When F is represented by equation (4.24), problem P(4.19) can be written as

P(4.20) min q

subject to

$$\max_{c\in \mathrm{cl}\,(C)_{1-\alpha^0}} f(c_i)\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + g(c_i) \leq q,\ i = 1, 2, \ldots, n\,,$$

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,,$$

$$\boldsymbol{x} \geq 0\,.$$

Let $Q = \{\varphi(r) \mid r \in \mathbf{R}\} \subseteq \mathbf{R}$ and $\varphi^{-1} : Q \to \mathbf{R}$ be the inverse function of φ. We have

$$\frac{\partial \varphi^{-1}(F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}))}{\partial c_j} = \begin{cases} f(c_i)x_j \leq 0, & \text{if } i \neq j, \\ f'(c_i)\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + g'(c_i) + f(c_i)x_i \geq 0, & \text{if } i = j. \end{cases}$$

Hence, problem P(4.20) can be reduced to the following linear programming problem:

P(4.21) min q

subject to

$$f(c_i^{\mathrm{R}}(1-\alpha^0))\left(\sum_{\substack{j=1\\ j\neq i}}^{n} c_j^{\mathrm{L}}(1-\alpha^0)x_j + c_i^{\mathrm{R}}(1-\alpha^0)x_i\right) \leq q - g(c_i^{\mathrm{R}}(1-\alpha^0)),\ i = 1,2,\ldots,n\,,$$

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,,$$

$$\boldsymbol{x} \geq 0\,.$$

Problem P(4.21) has $(n+1)$ constraints excluding non-negativity constraints. Therefore, even if the simplex method is applied, $(n+1)$ variables can take positive values. This means that problem P(4.21) can yield a diversified investment solution.

Table 4.1 Examples of f, g and φ

No.	Name	$f(r)$	$g(r)$	$\varphi(r)$
1	minimax regret	-1	r	r
2	minimax regret rate	$-\dfrac{1}{1+r}$	$\dfrac{r}{1+r}$	r
3	linear combination of 1 and 2 ($\kappa_1, \kappa_2 \geq 0$)	$-\dfrac{\kappa_1(1+r)+\kappa_2}{1+r}$	$\dfrac{\kappa_1 r^2 + (\kappa_1+\kappa_2)r}{1+r}$	r

Before describing numerical examples, we show some meaningful combinations of f, g and α in Table 4.1. The first one is the conventional minimax regret model [61] where the worst regret of x under a return rate vector $\boldsymbol{c}$ is defined by the difference between the optimal total return rate with respect to $\boldsymbol{c}$ and the obtained total return rate $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$. The second one is a regret rate model which is equivalent with the achievement rate model [62]. In the regret rate model, the worst regret of x under a return rate vector $\boldsymbol{c}$ is defined by the ratio of the difference between the optimal total return rate with respect to $\boldsymbol{c}$ and the obtained total return rate $\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}$ to the optimal total income rate. Note that, because of the negativity of $f(r)$, for all $r \in [L, R]$, we cannot consider the ratio to the optimal total return rate. The third one is the non-negative linear combination of the first and second ones.

4.2.4 Numerical Illustration

In order to see what solution is obtained from problem P(4.21), three numerical examples given by Inuiguchi and Tanino [63] are shown in this subsection. For comparison, not only the proposed approach but also a stochastic and the previous possibilistic programming approaches are applied to each example. In these examples, we have five assets whose possible return rate ranges are known as fuzzy sets having the following type of membership function:

$$\mu_{C_i}(r) = \exp\left(-\frac{(r - c_i^{\mathrm{c}})^2}{w_i}\right), \tag{4.25}$$

where c_i^{c} and w_i show the center and the spread.

As the corresponding probability distribution p_{C_i}, we consider the following normal distribution $\mathrm{N}(c_i^{\mathrm{c}}, \sqrt{w_i/2})$:

$$p_{C_i}(r) = \frac{1}{\sqrt{\pi w_i}} \exp\left(-\frac{(r - c_i^{\mathrm{c}})^2}{w_i}\right).$$

We assume the independence among return rates. The joint membership function is given as in equation (4.14) while the joint probability distribution is given as

$$p_{\boldsymbol{C}}(\boldsymbol{c}) = \prod_{1=1}^{n} p_{C_i}(c_i).$$

Since p_{C_i} is a normal distribution, $p_{\boldsymbol{C}}$ becomes a multivariate normal distribution with a mean vector $m = (c_1^{\mathrm{c}}, c_2^{\mathrm{c}}, \ldots, c_n^{\mathrm{c}})$ and a covariance matrix

$$V = \begin{pmatrix} \frac{w_1}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{w_2}{2} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & \frac{w_n}{2} \end{pmatrix}.$$

In the following examples, we assume that functions f, g and α are defined by those in the first row of Table 4.1, i.e., the conventional minimax regret model.

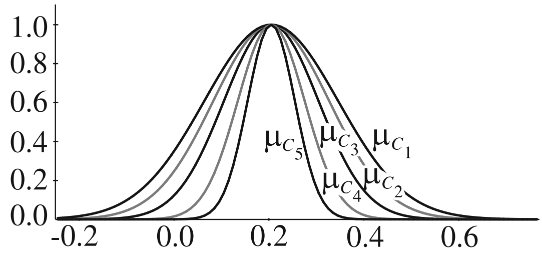


Fig. 4.3 Distribution of membership functions corresponding to five assets for Example 4.1

Table 4.2 Solutions for Example 1

Approach	x_1	x_2	x_3	x_4	x_5
Markowitz	0.061225	0.081633	0.122449	0.244898	0.489796
fractile	0	0	0	0	1
modality	0	0	0	0	1
spread	0	0	0	0	1
regret	0.25	0.211325	0.146446	0	0.392229

Example 4.1. *Let us consider five assets whose rate of return are given by membership functions depicted in Fig. 4.3. All c_i^c's are equal to 0.2 but w_i decreases as i increases:*

$$w_1 = 0.04,\ w_2 = 0.03,\ w_3 = 0.02,\ w_4 = 0.01 \quad and \quad w_5 = 0.005.$$

The optimal solutions to problems P(4.3), P(4.6), P(4.9), P(4.14) and P(4.21) are obtained as shown in Table 4.2 with setting $z^0 = 0.18$ and $\alpha^0 = 0.9$. Whereas, we obtain a concentrated investment solution on the 5th asset by the previous possibilistic programming approaches, i.e., the fractile optimization approach, the modality optimization approach and the spread minimization approach, we obtain a diversified investment solution on the 1st, 2nd, 3rd and 5th assets by the proposed possibilistic programming approach. However, the solution is rather different from the diversified investment solution of the Markowitz (stochastic programming) model P(4.3). The reason is that an asset with a small variance gathers the investment rate around it since minimization of the variance which is not related to the original objective, maximization of the total return rate, is adopted by the Markowitz model. This property of the Markowitz model will also be observed in Example 3.

Example 4.2. *Let us consider five assets whose rate of return are given by membership functions depicted in Fig. 4.4. All w_i's are equal to 0.02 but c_i^c decreases as i increases:*

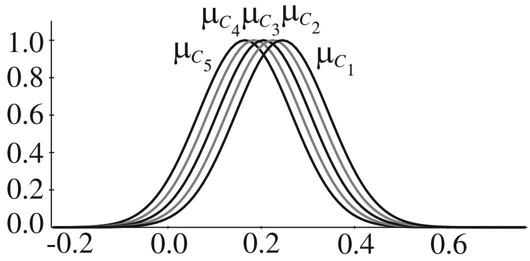


Fig. 4.4 Distribution of membership functions corresponding to five assets for Example 4.2

Table 4.3 Solutions for Example 2

Approach	x_1	x_2	x_3	x_4	x_5
Markowitz	0.200003	0.200002	0.200002	0.199997	0.199996
Fractile	1	0	0	0	0
Modality	1	0	0	0	0
Spread	1	0	0	0	0
Regret	0.293198	0.246599	0.2	0.153401	0.106802

$$c_1^c = 0.24,\ c_2^c = 0.22,\ c_3^c = 0.2,\ c_4^c = 0.18 \quad and \quad c_5^c = 0.16.$$

Table 4.3 shows the optimal solutions by Markowitz (stochastic programming), fractile, modality optimization, spread minimization and the conventional minimax regret approaches with setting $z^0 = 0.18$ *and* $\alpha^0 = 0.9$. *We can see that a diversified investment solution is not obtained by the previous possibilistic programming approaches, i.e., the fractile optimization approach, the modality optimization approach and the spread minimization approach, but by the proposed minimax regret approach. Markowitz model solution has almost equal investment rate proportions, whereas the proposed minimax approach solution claims a decreasing proportion going from right to left in Fig. 4.4 which corresponds to the decrease of the return rates.*

Table 4.4 Solutions for Example 3

Approach	x_1	x_2	x_3	x_4	x_5
Markowitz	0.192775	0.255173	0.232509	0.131889	0.187654
Fractile	0	1	0	0	0
Modality	1	0	0	0	0
Spread	0	0.42857	0	0.57143	0
Regret	0.408018	0.306634	0.252826	0.032522	0

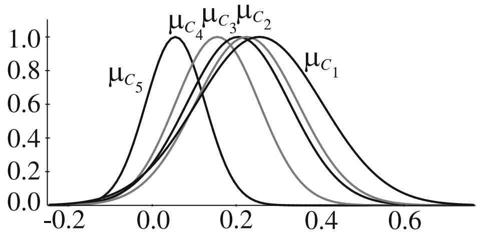


Fig. 4.5 Distribution of membership functions corresponding to five assets for Example 4.3

Example 4.3. *Let us consider five assets whose rate of return are given by membership functions depicted in Fig. 4.5. The parameters c_i^c's and w_i's are defined as*

$$c_1^c = 0.25,\ c_2^c = 0.22,\ c_3^c = 0.2,\ c_4^c = 0.15,\ c_5^c = 0.05,$$
$$w_1 = 0.0225,\ w_2 = 0.015,\ w_3 = 0.015,\ w_4 = 0.01 \quad \text{and} \quad w_5 = 0.005.$$

The larger c_i^c is, the larger is w_i. The 5th asset with $c_5^c = 0.05$ and $w_5 = 0.005$, intuitively speaking, seems to be inferior since it has the lowest return rate and is set apart from all the others on the critical edge close to zero. The optimal solutions by Markowitz, fractile, modality optimization, spread minimization and minimax regret approaches are obtained as shown in Table 4.4 with setting $z^0 = 0.18$ and $\alpha^0 = 0.9$. By the proposed minimax regret approach, we got a diversified investment solution but not by the other possibilistic programming approaches. The solution obtained from the proposed minimax regret approach does not support investment in the 5th asset but the Markowitz model solution does. In this case, the solution to problem P(4.21) seems to be better than that to problem P(4.3) since it is following the return rate pattern.

4.3 Portfolio Selection Using Interactive Coefficients

Another way to obtain a diversified investment solution is to introduce the interaction among return rates. The interaction stands for a mutual relation among rate of return of assets, e.g., 'an asset has a tendency to take a high rate of return if rate of return of certain assets are small', 'the sum of rate of return of several assets are more or less stable around 0.5', and so on. Until the previous section, we assumed the non-interaction among return rates. This implies that the possible range of the rate of return of an asset does not change with a realization of the rate of return of any other asset. Therefore, the investor is not motivated to distribute the fund to many assets. However, if rate of return of two assets have negative relations, i.e., one is high when the other is low, the investor may be motivated to distribute the fund to those assets in order to reduce risk. The introduction of interaction among rate of return would play a key role in possibilistic portfolio selection problem. We note that by the introduction of interaction, the necessity measure has a property to motivate the investor to distribute the fund to many assets, i.e., the necessity measure has the property described in Section 4.2.3.

However, the introduction of general interaction makes the reduced problem complex. In this subsection, we describe a few models to treat the interaction among rate of return without great loss of the tractability of the reduced problem. We describe the reduced problems of the necessity fractile optimization model P(4.5) and the minimax regret model P(4.17) with $F(r_1, r_2) = -r_1 - r_2$.

4.3.1 Scenario Decomposed Fuzzy Numbers

The rate of return of assets are often influenced by the economic situation. Then the estimated rate of return of assets can be different by the economic situation. This kind of the estimated rate of return can be represented by scenario decomposed fuzzy numbers proposed by Inuiguchi and Tanino [64]. In this approach, the possible ranges of uncertain parameters which depend on the situation are expressed by fuzzy if-then rules.

We may have a vague knowledge about the possible range of $\boldsymbol{\gamma}$ as the following k fuzzy if-then rules:

$$\text{if } s = s_k \text{ then } \boldsymbol{\gamma} \in \boldsymbol{C}^k,\ k = 1, 2, \ldots, u, \tag{4.26}$$

where s is a variable taking a value from $\{s_1, s_2, \ldots, s_u\}$. s is called a *scenario variable* and showing the situation. $\boldsymbol{C}^k = (C_1^k, C_2^k, \ldots, C_n^k)^{\mathrm{T}}$ is a vector of non-interactive fuzzy numbers. Namely, $\boldsymbol{C}^k$ has a membership function,

$$\mu_{\boldsymbol{C}^k}(\boldsymbol{c}) = \min\left(\mu_{C_1^k}(c_1), \mu_{C_2^k}(c_2), \ldots, \mu_{C_n^k}(c_n)\right),$$

and C_j^k is a fuzzy number such that $[C_j^k]_h = \{r \mid \mu_{C_j^k}(r) \geq h\}$ is a bounded closed interval, where $\mu_{C_j^k}$ is a membership function of a fuzzy number C_j^k. The body of rules (4.26) shows a fuzzy relation between scenario variable s and possible range of uncertain vector $\boldsymbol{\gamma}$. Namely, when $s = s_k$, the possible range of $\boldsymbol{\gamma}$ is $\boldsymbol{C}^k$.

For example, we may have knowledge,

- if economic situation s is s_1 then the return rate vector $\boldsymbol{\gamma}$ is in a possible range $\boldsymbol{C}^1$,
- if economic situation s is s_2 then the return rate vector $\boldsymbol{\gamma}$ is in a possible range $\boldsymbol{C}^2$,
- if economic situation s is s_3 then the return rate vector $\boldsymbol{\gamma}$ is in a possible range $\boldsymbol{C}^3$.

(4.27)

When we obtain the estimated range of scenario variable s as a fuzzy set S showing a possible range of s, the estimated fuzzy set $\boldsymbol{C}$ is obtained as

$$\mu_{\boldsymbol{C}}(\boldsymbol{c}) = \max_{k=1,2,\ldots,u} \min\left(\mu_S(s_k), \mu_{\boldsymbol{C}^k}(\boldsymbol{c}),\right) \tag{4.28}$$

where μ_S is a membership function of S. This equation is based on the fuzzy reasoning proposed by Zadeh [124].

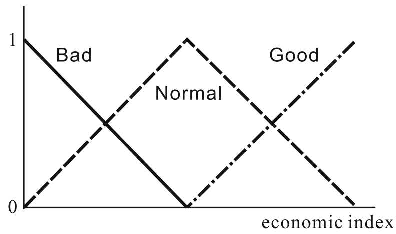


Fig. 4.6 An example of fuzzy partition

Inuiguchi and Tanino [64] considered a continuous scenario variable. In the continuous scenario variable case, the knowledge can be represented by a set of fuzzy if-then rules, 'if s is in fuzzy set S^k then γ is in fuzzy set C^k'. For example, to make (4.27) be a set of fuzzy rules, we may partition the range of economic index into three fuzzy sets shown in Fig. 4.6.

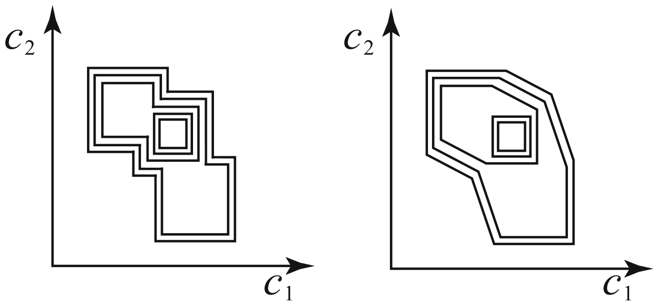
Let μ_{S^k} be the membership function of fuzzy set S^k in the antecedent part of fuzzy if-then rules. For $s = \bar{s}$, the estimated range $C_j(\bar{s})$ of the return rate of j-th asset is defined by

$$C_j(\bar{s}) = \frac{\sum_{k=1}^{u} \mu_{S^k}(\bar{s}) C_j^k}{\sum_{k=1}^{u} \mu_{S^k}(\bar{s})}. \tag{4.29}$$

The extension principle is applied to calculate $C_j(\bar{s})$. Let $\boldsymbol{C}(\bar{s}) = (C_1(\bar{s}), C_2(\bar{s}), \ldots, C_n(\bar{s}))^{\mathrm{T}}$. The estimated fuzzy set $\boldsymbol{C}$ under a fuzzy set S showing a possible realizations of s is obtained as

$$\mu_{\boldsymbol{C}}(\boldsymbol{c}) = \sup_{s} \min\left(\mu_S(s), \mu_{\boldsymbol{C}(s)}(\boldsymbol{c})\right). \tag{4.30}$$

Level curves of the membership function of scenario decomposed fuzzy numbers $\boldsymbol{C}$ defined by equation (4.28) with $n = 2$ is depicted in Fig. 4.7(a). In equation (4.26), we consider a discrete scenario variable s. Level curves of the membership function of scenario decomposed fuzzy numbers $\boldsymbol{C}$ defined by equation (4.30) with a continuous scenario variable when $n = 2$ is illustrated



(a) Discrete scenario variable (b) Continuous scenario variable

Fig. 4.7 Lever curves of the membership functions of scenario decomposed fuzzy numbers

in Fig. 4.7(b). In this subsection, we concentrate on the continuous scenario variable case.

The fuzzy set whose membership function is defined by equations (4.26) and (4.30) is called scenario decomposed fuzzy numbers.

Now let us investigate the possible range of a linear function value $\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}$ with scenario decomposed fuzzy numbers $\boldsymbol{C}$ with membership function defined by equation (4.30). Let $Y(\boldsymbol{x})$ and $Y^k(\boldsymbol{x})$ be fuzzy sets defined by the following membership functions:

$$\mu_{Y(\boldsymbol{x})}(y) = \sup\left\{\mu_{\boldsymbol{C}}(\boldsymbol{c}) \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = y\right\}, \tag{4.31}$$

$$\mu_{Y^k(\boldsymbol{x})}(y) = \sup\left\{\mu_{\boldsymbol{C}^k}(\boldsymbol{c}) \mid \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = y\right\}. \tag{4.32}$$

Namely, $Y(\boldsymbol{x})$ shows the overall possible range of $\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}$ while $Y^k(\boldsymbol{x})$ shows the possible range of $\boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{x}$ when the possible range of $\boldsymbol{\gamma}$ is given by $\boldsymbol{C}^k$.

Because the linearity of function is preserved in the extension principle, We obtain the following relation between $Y(\boldsymbol{x})$ and $Y^k(\boldsymbol{x})$:

$$\mu_{Y(\boldsymbol{x})}(y) = \sup_{s}\min\left(\mu_S(s), \sup_{\boldsymbol{r}:\boldsymbol{\mu}(s)^{\mathrm{T}}\boldsymbol{r}=y}\min\left(\mu_{Y^1(\boldsymbol{x})}(r_1), \mu_{Y^2(\boldsymbol{x})}(r_2), \ldots, \mu_{Y^u(\boldsymbol{x})}(r_u)\right)\right), \tag{4.33}$$

where $\boldsymbol{r} = (r_1, r_2, \ldots, r_u)^{\mathrm{T}}$ and

$$\boldsymbol{\mu}(s) = (\mu_1(s), \mu_2(s), \ldots, \mu_u(s))^{\mathrm{T}} = \frac{(\mu_{S^1}(s), \mu_{S^2}(s), \ldots, \mu_{S^u}(s))^{\mathrm{T}}}{\sum_{k=1}^{u}\mu_{S^k}(s)}. \tag{4.34}$$

From equations (4.8), (4.29) and the non-negativity of $\boldsymbol{x}$, we obtain

$$N_{Y(\boldsymbol{x})}([z,+\infty)) \geq \alpha^0 \Leftrightarrow \mathrm{cl}(Y(\boldsymbol{x}))_{1-\alpha^0} \subseteq [z,+\infty)$$
$$\Leftrightarrow \mathrm{cl}(Y^k(\boldsymbol{x}))_{1-\alpha^0} \subseteq [z,+\infty),\ \forall s \text{ such that } \mu_S(s) > 1-\alpha^0$$
$$\Leftrightarrow \sum_{j=1}^{n}\sum_{k=1}^{u} \mu_k(s)\bar{c}^{\mathrm{L}}_{jk}(1-\alpha^0)x_j \geq z,\ \forall s \text{ such that } \mu_S(s) > 1-\alpha^0, \quad (4.35)$$

where we define $\mathrm{cl}(C_j^k)_\alpha = [\bar{c}^{\mathrm{L}}_{jk}(\alpha), \bar{c}^{\mathrm{R}}_{jk}(\alpha)]$, $k = 1,2,\ldots,u$ and from equation (4.34), we hvae $\mu_k(s) = \mu_{S^k}(s)/\sum_{j=1}^{u}\mu_{S^j}(s)$.

The necessity fractile optimization model P(4.5) is reduced to the following programming problem:

$$\begin{aligned}
\textbf{P(4.22)}\quad & \max\ z \\
& \text{subject to} \\
& \sum_{j=1}^{n}\sum_{k=1}^{u} \mu_k(s)\bar{c}^{\mathrm{L}}_{jk}(1-\alpha^0)x_j \geq z,\ \forall s \text{ such that } \mu_S(s) > 1-\alpha^0, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1, \\
& \boldsymbol{x} \geq 0.
\end{aligned}$$

When the range of scenario variable s is a subset of real line $\mathbf{R}$, the antecedent fuzzy sets S^k, $k = 1,2,\ldots,u$ are fuzzy numbers satisfying

$$\sum_{k=1}^{u} \mu_{S^k}(s) = 1,\ \forall s, \quad (4.36)$$

and S is also a fuzzy number, problem P(4.22) is reduced to the following linear programming problem [64]:

$$\begin{aligned}
\textbf{P(4.23)}\quad & \max\ z \\
& \text{subject to} \\
& \sum_{j=1}^{n} \bar{c}^{\mathrm{L}}_{jk}(1-\alpha^0)x_j \geq z,\ \forall k \text{ such that } [S]_1 \cap [S^k]_1 \neq \emptyset, \\
& \sum_{j=1}^{n}\sum_{k=1}^{u} \mu_k(\bar{s}^{\mathrm{L}}(1-\alpha^0))\bar{c}^{\mathrm{L}}_{jk}(1-\alpha^0)x_j \geq z, \\
& \sum_{j=1}^{n}\sum_{k=1}^{u} \mu_k(\bar{s}^{\mathrm{R}}(1-\alpha^0))\bar{c}^{\mathrm{L}}_{jk}(1-\alpha^0)x_j \geq z, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1, \\
& \boldsymbol{x} \geq 0,
\end{aligned}$$

where $\bar{s}^{\mathrm{L}}(\alpha)$ and $\bar{s}^{\mathrm{R}}(\alpha)$ are defined by $\mathrm{cl}(S)_\alpha = [\bar{s}^{\mathrm{L}}(\alpha), \bar{s}^{\mathrm{R}}(\alpha)]$. Because problem P(4.23) has several constraints, some of variables x_j, $j = 1,2,\ldots,n$ may take positive values.

Now we describe the minimax regret model. We again assume that the range of scenario variable s is a subset of real line $\mathbf{R}$, the antecedent fuzzy sets S^k, $k = 1, 2, \ldots, u$ are fuzzy numbers satisfying equation (4.36), and S is also a fuzzy number.

Because C is the scenario decomposed fuzzy numbers, problem P(4.19) is reduced to

P(4.24) $\min\ q$

subject to

$$\max_{\boldsymbol{c} \in \mathrm{cl}\ (C^k)_{1-\alpha^0}} F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}) \le q,\ \ i = 1, 2, \ldots, n,\ \ k \text{ such that } [S]_1 \cap [S^k]_1 \ne \emptyset\,,$$

$$\max_{\boldsymbol{c} \in \mathrm{cl}\ (C(\bar{s}^{\mathrm{L}}(1-\alpha^0)))_{1-\alpha^0}} F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}) \le q,\ \ i = 1, 2, \ldots, n\,,$$

$$\max_{\boldsymbol{c} \in \mathrm{cl}\ (C(\bar{s}^{\mathrm{R}}(1-\alpha^0)))_{1-\alpha^0}} F(c_i, \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x}) \le q,\ \ i = 1, 2, \ldots, n\,,$$

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,,$$

$$\boldsymbol{x} \ge 0\,.$$

By the same discussion as in Section 4.2.3 when F is defined by equation (4.24), problem P(4.24) is reduced to the following linear programming problem:

P(4.25) $\min\ q$

subject to

$$f(c_{ik}^{\mathrm{R}}(1-\alpha^0))\left(\sum_{\substack{j=1\\ j\ne i}}^{n} c_{jk}^{\mathrm{L}}(1-\alpha^0)x_j + c_{ik}^{\mathrm{R}}(1-\alpha^0)x_i\right)$$

$$\le q - g(c_{ik}^{\mathrm{R}}(1-\alpha^0)),\ \ i = 1, 2, \ldots, n,\ \ k \text{ such that } [S]_1 \cap [S^k]_1 \ne \emptyset\,,$$

$$f\left(\boldsymbol{\mu}(\bar{s}^{\mathrm{L}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)\right)$$

$$\times\left(\sum_{\substack{j=1\\ j\ne i}}^{n} \boldsymbol{\mu}(\bar{s}^{\mathrm{L}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_j^{\mathrm{L}}(1-\alpha^0)x_j + \boldsymbol{\mu}(\bar{s}^{\mathrm{L}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)x_i\right)$$

$$\le q - g\left(\boldsymbol{\mu}(\bar{s}^{\mathrm{L}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)\right),\ \ i = 1, 2, \ldots, n\,,$$

$$f\left(\boldsymbol{\mu}(\bar{s}^{\mathrm{R}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)\right)$$

$$\times\left(\sum_{\substack{j=1\\ j\ne i}}^{n} \boldsymbol{\mu}(\bar{s}^{\mathrm{R}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_j^{\mathrm{L}}(1-\alpha^0)x_j + \boldsymbol{\mu}(\bar{s}^{\mathrm{R}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)x_i\right)$$

$$\leq q - g\left(\boldsymbol{\mu}(\bar{s}^{\mathrm{R}}(1-\alpha^0))^{\mathrm{T}}\bar{\boldsymbol{c}}_i^{\mathrm{R}}(1-\alpha^0)\right),\ i = 1,2,\ldots,n\,,$$
$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,,$$
$$\boldsymbol{x} \geq 0\,,$$

where $\bar{\boldsymbol{c}}_j^{\mathrm{L}}(\alpha) = (c_{j1}^{\mathrm{L}}(\alpha), c_{j2}^{\mathrm{L}}(\alpha), \ldots, c_{ju}^{\mathrm{L}}(\alpha))^{\mathrm{T}}$ and $\bar{\boldsymbol{c}}_j^{\mathrm{R}}(\alpha) = (c_{j1}^{\mathrm{R}}(\alpha), c_{j2}^{\mathrm{R}}(\alpha), \ldots, c_{ju}^{\mathrm{R}}(\alpha))^{\mathrm{T}}$. This problem has more constraints than problem P(4.23). Then, roughly speaking, it is more probable that the optimal solution to problem P(4.25) suggests a diversified investment.

The reduced problem of the conventional minimax regret model with $F(r_1, r_2) = r_1 - r_2$ is obtained by substituting $f(r) = -1$ and $g(r) = r$ in problem P(4.25).

Remark 4.1. *When the scenario variable obeys a probability distribution under fuzzy if-then rules, the return rate becomes a fuzzy random number [73]. On the other hand, when the fuzzy number C^k is replaced with a probability distribution, the return rate obeys a mixture model [10].*

4.3.2 Oblique Fuzzy Vector

From the historical data, we may find a vague knowledge about a linear function value of rate of return of several assets and the differences of two uncertain values, e.g., $\gamma_1 + 2\gamma_2 + \gamma_3$ is about 1.3, $|\gamma_4 - \gamma_5|$ is approximately 0.1, and so on. If we have only n independent pieces of vague knowledge about the linear function values of return rates of assets, we can apply oblique fuzzy vector [60] to represent the possible range of return rate vector.

Oblique fuzzy vectors are proposed by Inuiguchi, Ramík and Tanino [60] and each of them can express n independent pieces of vague knowledge about the linear function values of uncertain values. A non-singular matrix shows the interaction among uncertain parameters in an oblique fuzzy vector as a covariance matrix shows in a multivariate normal distribution.

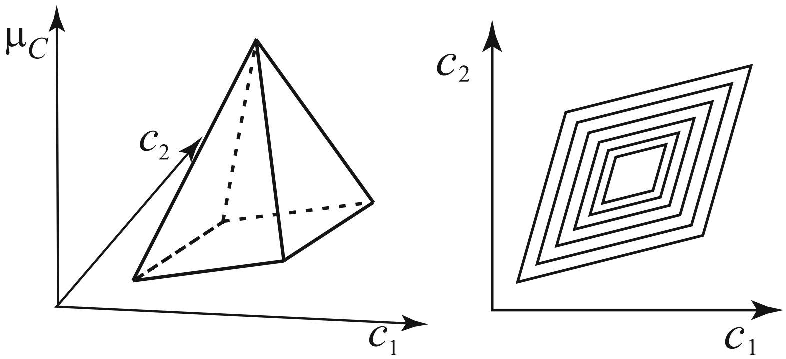
An oblique fuzzy vector $\boldsymbol{C}$ is defined by the following membership function,

$$\mu_{\boldsymbol{C}}(\boldsymbol{c}) = \min_{j=1,2,\ldots,n} \mu_{B_j}(\boldsymbol{d}_j^{\mathrm{T}}\boldsymbol{c}),$$

where μ_{B_j} is a membership function of an L-L fuzzy number $B_j = (b_j^{\mathrm{L}}, b_j^{\mathrm{R}}, \beta_j^{\mathrm{L}}, \beta_j^{\mathrm{R}})_{LL}$ and $\boldsymbol{d}_j$, $j = 1,2,\ldots,n$ are vectors such that $D = (\boldsymbol{d}_1, \boldsymbol{d}_2, \ldots, \boldsymbol{d}_n)^{\mathrm{T}}$ be a non-singular real-valued $n \times n$ matrix. An L-L fuzzy number $B_j = (b_j^{\mathrm{L}}, b_j^{\mathrm{R}}, \beta_j^{\mathrm{L}}, \beta_j^{\mathrm{R}})_{LL}$ can be characterized by the following membership function:

$$\mu_{B_j}(r) = \begin{cases} L\left(\dfrac{b_j^{\mathrm{L}} - r}{\beta_j^{\mathrm{L}}}\right), & \text{if } r < b_j^{\mathrm{L}}, \\ 1, & \text{if } b_j^{\mathrm{L}} \le r \le b_j^{\mathrm{R}}, \\ L\left(\dfrac{r - b_j^{\mathrm{R}}}{\beta_j^{\mathrm{R}}}\right), & \text{if } r > b_j^{\mathrm{R}}, \end{cases}$$

where we assume $b_j^{\mathrm{L}} \le b_j^{\mathrm{R}}$, $\beta_j^{\mathrm{L}} > 0$ and $\beta_j^{\mathrm{R}} > 0$. $L : [0, +\infty) \to [0, 1]$ is a reference function defined earlier. Namely, an oblique fuzzy vector can be obtained from n pieces of knowledge '$\boldsymbol{d}_j^{\mathrm{T}}\boldsymbol{c}$ takes a value in a fuzzy number B_j', $j = 1, 2, \ldots, n$, where $\boldsymbol{d}_j$, $j = 1, 2, \ldots, n$ should be linearly independent.



(a) 3D image (b) Level curves

Fig. 4.8 The membership function of an oblique fuzzy vector

An example of an oblique fuzzy vector when $n = 2$ is given in Fig. 4.8. Unlike non-interactive fuzzy numbers, the level curves of an oblique fuzzy vector are neither always rectangle nor parallel to coordinate axes.

It is shown that a linear function value $Y(\boldsymbol{x})$ of an oblique fuzzy vector can be calculated easily. $Y(\boldsymbol{x})$ is defined by the membership function (4.31) with oblique fuzzy vector C. Inuiguchi, Ramík and Tanino [60] obtained the following useful result:

$$\mathrm{cl}(Y(\boldsymbol{x}))_\alpha = \left[\sum_{j:k_j(\boldsymbol{x})\ge 0} \bar{b}_j^{\mathrm{L}}(\alpha)k_j(\boldsymbol{x}) + \sum_{j:k_j(\boldsymbol{x})<0} \bar{b}_j^{\mathrm{R}}(\alpha)k_j(\boldsymbol{x}), \right.$$
$$\left. \sum_{j:k_j(\boldsymbol{x})\ge 0} \bar{b}_j^{\mathrm{R}}(\alpha)k_j(\boldsymbol{x}) + \sum_{j:k_j(\boldsymbol{x})<0} \bar{b}_j^{\mathrm{L}}(\alpha)k_j(\boldsymbol{x}) \right], \ \forall \alpha \in [0, 1), \tag{4.37}$$

where $k_j(\boldsymbol{x})$, $j = 1, 2, \ldots, n$ are defined as follows with d^*_{ij}, the (i, j) component of D^{-1};

$$k_j(\boldsymbol{x}) = \sum_{i=1}^{n} d^*_{ij} x_i.$$

$\bar{b}^{\mathrm{L}}_j(h)$ and $\bar{b}^{\mathrm{R}}_j(h)$ are defined by

$$\bar{b}^{\mathrm{L}}_j(h) = b^{\mathrm{L}}_j - \beta^{\mathrm{L}}_j L^*(h),$$
$$\bar{b}^{\mathrm{R}}_j(h) = b^{\mathrm{R}}_j - \beta^{\mathrm{R}}_j L^*(h),$$

where L^* is defined by equation (4.18). This result implies that the linear function values of an oblique fuzzy vector is an L-L fuzzy number [60].

We consider the necessity fractile optimization model P(4.5). Applying the result (4.37), the lower bound of $\mathrm{cl}(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{x})_{1-\alpha^0} = \mathrm{cl}(Y(\boldsymbol{x}))_{1-\alpha^0}$, i.e., $\boldsymbol{c}^{\mathrm{L}}(1-\alpha^0)^{\mathrm{T}}\boldsymbol{x}$ in problem P(4.6) is replaced with $\sum_{j:k_j(\boldsymbol{x})\geq 0} \bar{b}^{\mathrm{L}}_j(\alpha)k_j(\boldsymbol{x}) + \sum_{j:k_j(\boldsymbol{x})<0} \bar{b}^{\mathrm{R}}_j(\alpha)k_j(\boldsymbol{x})$, we obtain the following problem:

$$
\begin{aligned}
\textbf{P(4.26)}\quad & \max \sum_{j:k_j(\boldsymbol{x})\geq 0} \bar{b}^{\mathrm{L}}_j(1-h^0)k_j(\boldsymbol{x}) + \sum_{j:k_j(\boldsymbol{x})<0} \bar{b}^{\mathrm{R}}_j(1-h^0)k_j(\boldsymbol{x}) \\
& \text{subject to} \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \geq \boldsymbol{0}\,, \\
& k_j(\boldsymbol{x}) = \sum_{i=1}^{n} d^*_{ij} x_i\,.
\end{aligned}
$$

We have

$$\boldsymbol{k}(\boldsymbol{x}) = (k_1(\boldsymbol{x}), k_2(\boldsymbol{x}), \ldots, k_n(\boldsymbol{x}))^{\mathrm{T}} = D^{-\mathrm{T}}\boldsymbol{x}, \tag{4.38}$$

where $D^{-\mathrm{T}} = {D^{-1}}^{\mathrm{T}} = {D^{\mathrm{T}}}^{-1}$. From this fact, we introduce variable vectors $\boldsymbol{y}^+ = (y^+_1, y^+_2, \ldots, y^+_n)^{\mathrm{T}}$ and $\boldsymbol{y}^- = (y^-_1, y^-_2, \ldots, y^-_n)^{\mathrm{T}}$ such that

$$D^{-\mathrm{T}}\boldsymbol{x} = \boldsymbol{y}^+ - \boldsymbol{y}^-,\ {\boldsymbol{y}^+}^{\mathrm{T}}\boldsymbol{y}^- = 0,\ \boldsymbol{y}^+ \geq \boldsymbol{0},\ \boldsymbol{y}^- \geq \boldsymbol{0}. \tag{4.39}$$

From equations (4.38) and (4.39), we have $k_j(\boldsymbol{x}) = y^+_j$ if $k_j(\boldsymbol{x}) \geq 0$ and $k_j(\boldsymbol{x}) = -y^-_j$ if $k_j(\boldsymbol{x}) < 0$. Moreover, from the first equation of (4.39), we have $\boldsymbol{x} = D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-)$. Introducing those, we can prove problem P(4.26) is reduced to the following linear programming problem:

$$\begin{aligned}\mathbf{P(4.27)}\quad &\max \sum_{j=1}^{n} \bar{b}_j^{\mathrm{L}}(1-h^0)y_j^+ - \sum_{j=1}^{n} \bar{b}_j^{\mathrm{R}}(1-h^0)y_j^- \\ &\text{subject to} \\ &\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\ &\boldsymbol{x} \geq \boldsymbol{0}\,, \\ &\boldsymbol{x} = D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-)\,, \\ &\boldsymbol{y}^+ \geq \boldsymbol{0}\,, \\ &\boldsymbol{y}^- \geq \boldsymbol{0}\,,\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathbf{P(4.28)}\quad &\max \sum_{j=1}^{n} \bar{b}_j^{\mathrm{L}}(1-h^0)y_j^+ - \sum_{j=1}^{n} \bar{b}_j^{\mathrm{R}}(1-h^0)y_j^- \\ &\text{subject to} \\ &\boldsymbol{e}^{\mathrm{T}}D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-) = 1\,, \\ &D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-) \geq \boldsymbol{0}\,, \\ &\boldsymbol{y}^+ \geq \boldsymbol{0}\,, \\ &\boldsymbol{y}^- \geq \boldsymbol{0}\,.\end{aligned}$$

Note that a complementary condition $(\boldsymbol{y}^+)^{\mathrm{T}}\boldsymbol{y}^- = 0$ can be omitted in problem P(4.27) because we obtain a solution satisfying this condition easily from any optimal solution of problem P(4.27) without change of $\boldsymbol{x}$. It is shown in [60] that Bender's decomposition method can be applied to problem P(4.27). When $D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-)$ is non-negative for any $\boldsymbol{y}^- \geq \boldsymbol{0}$ and $\boldsymbol{y}^+ \geq \boldsymbol{0}$ such that $\boldsymbol{e}^{\mathrm{T}}D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-) = 1$, $D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-) \geq \boldsymbol{0}$ is never active. Therefore, in this case, only one y_i^+ or y_i^- may take a positive value at a basic feasible solution of problem P(4.28). However, because of $\boldsymbol{x} = D^{\mathrm{T}}(\boldsymbol{y}^+ - \boldsymbol{y}^-)$, some components of $\boldsymbol{x}$ may take positive values even in this case.

Now let us investigate a minimax regret model with $F(r_1, r_2) = r_1 - r_2$. From problem P(4.19), we consider linear function values

$$R_i(\boldsymbol{x}) = c_i - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} = \sum_{\substack{l=1\\ l\neq i}}^{n} c_l x_l + c_i(1-x_i). \tag{4.40}$$

In the same way as we did for $Y(\boldsymbol{x})$, we obtain

$$\mathrm{cl}(R_i(\boldsymbol{x}))_\alpha = \Bigg[\sum_{j:k_j^i(\boldsymbol{x})\geq 0} \bar{b}_j^{\mathrm{L}}(\alpha)k_j^i(\boldsymbol{x}) + \sum_{j:k_j^i(\boldsymbol{x})<0} \bar{b}_j^{\mathrm{R}}(\alpha)k_j^i(\boldsymbol{x}), \\ \sum_{j:k_j^i(\boldsymbol{x})\geq 0} \bar{b}_j^{\mathrm{R}}(\alpha)k_j^i(\boldsymbol{x}) + \sum_{j:k_j^i(\boldsymbol{x})<0} \bar{b}_j^{\mathrm{L}}(\alpha)k_j^i(\boldsymbol{x})\Bigg],\ \forall\alpha\in[0,1),$$

where we define

$$k_j^i(\boldsymbol{x}) = \sum_{\substack{l=1 \\ l \neq i}}^{n} d_{lj}^* x_l + d_{ij}^*(1 - x_i).$$

Introducing variable vectors $\boldsymbol{y}_i^+ = (y_{i1}^+, y_{i2}^+, \ldots, y_{in}^+)^{\mathrm{T}}$ and $\boldsymbol{y}_i^- = (y_{i1}^-, y_{i2}^-, \ldots, y_{in}^-)^{\mathrm{T}}$ such that

$$k_j^i(\boldsymbol{x}) = y_{ij}^+ - y_{ij}^-,\ y_{ij}^+ \cdot y_{ij}^- = 0,\ y_{ij}^+ \geq 0,\ y_{ij}^- \geq 0,\ j = 1, 2, \ldots, n,$$

we obtain the following reduced linear programming problem:

$$\begin{aligned}
\textbf{P(4.29)}\quad & \min\ q \\
& \text{subject to} \\
& \sum_{j=1}^{n} \bar{b}_j^{\mathrm{R}}(1 - h^0) y_{ij}^+ - \sum_{j=1}^{n} \bar{b}_j^{\mathrm{L}}(1 - h^0) y_{ij}^- \leq q,\ i = 1, 2, \ldots, n, \\
& x_i = 1 - \sum_{j=1}^{n} d_{ij}(y_{ij}^+ - y_{ij}^-),\ i = 1, 2, \ldots, n, \\
& x_l = \sum_{j=1}^{n} d_{lj}(y_{ij}^+ - y_{ij}^-),\ l = 1, 2, \ldots, n\ (l \neq i),\ i = 1, 2, \ldots, n, \\
& y_{ij}^+ \geq 0,\ y_{ij}^- \geq 0,\ i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, n, \\
& \boldsymbol{e}^{\mathrm{T}} \boldsymbol{x} = 1, \\
& \boldsymbol{x} \geq 0,
\end{aligned}$$

where d_{ij} is the (i, j)-component of D. This problem can also be solved by Bender's decomposition method. Because problem P(4.29) has many constraints, many components of $\boldsymbol{x}$ can take positive values so that it suggests a diversified investment solution.

4.3.3 Fuzzy Polytope

By oblique fuzzy vector, we can express n independent pieces of vague knowledge about linear function values of uncertain parameters. However, in the real-world, we may have more than n pieces of vague knowledge including vague knowledge about the ratio between two uncertain parameters. The ratio between two uncertain parameters cannot be expressed as a linear function of uncertain parameters. Therefore such a body of vague knowledge cannot be expressed well by an oblique fuzzy vector.

Inuiguchi and Tanino [65] introduced a fuzzy polytope to fuzzy linear programming problems. A fuzzy polytope can express more than n pieces of

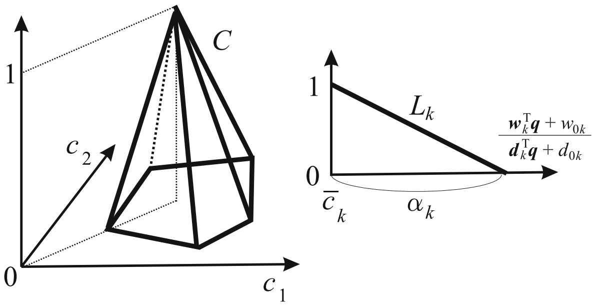


Fig. 4.9 The membership function of a fuzzy polytope

vague knowledge about linear fractional function values of uncertain parameters. Then the oblique fuzzy vector is a special case of the fuzzy polytope.

When $\boldsymbol{C}$ is a fuzzy polytope, its membership function is expressed as

$$\mu_{\boldsymbol{C}}(\boldsymbol{c}) = \min_{k=1,2,\ldots,v} L_k \left(\frac{\dfrac{\boldsymbol{w}_k^{\mathrm{T}} \boldsymbol{c} + w_{0k}}{\boldsymbol{d}_k^{\mathrm{T}} \boldsymbol{c} + d_{0k}} - \bar{q}_k}{\alpha_k} \right), \tag{4.41}$$

where $L_k : \mathbf{R} \to [0, 1]$, $k = 1, 2, \ldots, v$ are reference functions, i.e., upper semi-continuous non-increasing functions such that $L_k(0) = 1$ and $\lim_{r\to+\infty} L_k(r) = 0$. $\bar{q}_k$ is the most plausible value for the k-th linear fractional function value $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{\gamma} + w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{\gamma} + d_{0k})$. α_k shows the spread, i.e., to what extent the linear fractional function value $(\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{\gamma} + w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{\gamma} + d_{0k})$ possibly exceeds $\bar{q}_k$. The knowledge about the maximum possible shortage (α_l) of $(\boldsymbol{w}_l^{\mathrm{T}}\boldsymbol{\gamma}+w_{0l})/(\boldsymbol{d}_l^{\mathrm{T}}\boldsymbol{\gamma}+d_{0l})$ from $\bar{q}_l$ is treated as the knowledge of the maximum possible exceeds (α_l) of $(-\boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{\gamma} - w_{0k})/(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{\gamma} + d_{0k})$ from $-\bar{q}_k$. Fuzzy set $\boldsymbol{C}$ is assumed to be bounded, i.e., h-level sets $[\boldsymbol{C}]_h = \{\boldsymbol{c} \mid \mu_{\boldsymbol{C}}(\boldsymbol{c}) \geq h\}$ for all $h \in (0, 1]$ are bounded. Moreover, without loss of generality, we assume that $\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c} + d_{0k} > 0$ for all possible $\boldsymbol{c}$. Let $L_k^*(h) = \sup\{r \mid L_k(r) > h\}$ for $h \in [0, 1)$ and $L_k^*(h) = -\infty$ for $h = 1$.

Since a linear fractional function includes a sum, a difference, a linear function and a ratio, a fuzzy polytope is useful when we know possible ranges of a sum of uncertain variables, a difference between two uncertain variables, a linear function values of uncertain variables and a ratio between two uncertain variables. The membership function of a fuzzy polytope when $n = 2$ is depicted in Fig. 4.9.

Because of equation (4.41), we have

$$\begin{aligned} \mathrm{cl}(\boldsymbol{C})_h &= \{\boldsymbol{c} \mid \boldsymbol{w}_k^{\mathrm{T}}\boldsymbol{c} + w_{0k} \leq (\bar{q}_k + \alpha_k L_k^*(h))(\boldsymbol{d}_k^{\mathrm{T}}\boldsymbol{c} + d_{0k}),\ k = 1, 2, \ldots, v\} \\ &= \{\boldsymbol{c} \mid \boldsymbol{wd}_k^*(h)^{\mathrm{T}}\boldsymbol{c} \leq -wd_{0k}^*(h),\ k = 1, 2, \ldots, v\}, \end{aligned} \tag{4.42}$$

where $\boldsymbol{wd}_k^*(h) = \boldsymbol{w}_k - (\bar{q}_k + \alpha_k L_k^*(h))\boldsymbol{d}_k$ and $wd_{0k}^*(h) = w_{0k} - (\bar{q}_k + \alpha_k L_k^*(h))d_{0k}$. Since $[\boldsymbol{C}]_h \subseteq \mathbf{R}^n$ is bounded, from (4.42), we know that $v > n$ and that $[\boldsymbol{C}]_h$ and $\mathrm{cl}(\boldsymbol{C})_h$ are polytopes for all $h \in [0, 1)$.

Using equation (4.16), problem P(4.5) is reduced to the following semi-infinite programming problem:

$$\begin{aligned} \mathbf{P(4.30)}\quad & \max\ z \\ & \text{subject to} \\ & \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \geq z,\ \forall \boldsymbol{c} \in \mathrm{cl}(\boldsymbol{C})_{1-h^0}, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Together with equation (4.42), problem P(4.30) can be solved by the following relaxation procedure.

Solution Algorithm for Problem P(4.30)

Step 1: Select $\boldsymbol{x}^0$ satisfying $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}^0 = 1$, $\boldsymbol{x}^0 \geq \boldsymbol{0}$. Let $z^0 = \infty$ and $l = 0$.

Step 2: Solve a linear programming problem

$$\begin{aligned} \mathbf{P(4.31)}\quad & \min\ \boldsymbol{x}^{0\mathrm{T}}\boldsymbol{c} \\ & \text{subject to} \\ & \boldsymbol{wd}_k^*(h)^{\mathrm{T}}\boldsymbol{c} \leq -wd_{0k}^*(h),\ k = 1, 2, \ldots, v. \end{aligned}$$

Let $\hat{\boldsymbol{c}}$ be an obtained optimal solution to problem P(4.31).

Step 3: If $\hat{\boldsymbol{c}}^{\mathrm{T}}\boldsymbol{x}^0 < z^0$ then update $l = l + 1$ and let $\boldsymbol{c}_l = \hat{\boldsymbol{c}}$. Otherwise, we terminate the algorithm and obtain an optimal solution $\boldsymbol{x}^0$ to problem P(4.30).

Step 4: Solve a linear programming problem,

$$\begin{aligned} \mathbf{P(4.32)}\quad & \max\ z \\ & \text{subject to} \\ & \boldsymbol{c}_w^{\mathrm{T}}\boldsymbol{x} \geq z,\ w = 1, 2, \ldots, l, \\ & \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1, \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

Let $(\boldsymbol{x}^{0\mathrm{T}}, z^0)^{\mathrm{T}}$ be an obtained optimal solution. Return to Step 2.

In the algorithm described above, we solve two kinds of linear programming problems; therefore, we can solve problem P(4.30) using linear programming techniques only.

Let us investigate the minimax regret model with $F(r_1, r_2) = r_1 - r_2$. From problem P(4.19), the problem is reduced to the following semi-infinite programming problem:

$$
\begin{aligned}
\mathbf{P(4.32)}\quad & \min\ z \\
& \text{subject to} \\
& c_i - \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \le z,\ \forall \boldsymbol{c} \in \mathrm{cl}(\boldsymbol{C})_{1-h^0},\ i = 1,2,\dots,n\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \ge \boldsymbol{0}\,.
\end{aligned}
$$

Together with equation (4.42), problem P(4.32) can also be solved by the following relaxation procedure.

Solution Algorithm for Problem P(4.32)

Step 1: Select $\boldsymbol{x}^0$ satisfying $\boldsymbol{e}^{\mathrm{T}}\boldsymbol{x}^0 = 1$, $\boldsymbol{x}^0 \ge \boldsymbol{0}$. Let $q^0 = -\infty$ and $l_i = 0$, $i = 1,2,\dots,n$.

Step 2: For $i = 1,2,\dots,n$, solve linear programming problems

$$
\begin{aligned}
\mathbf{P(4.33)}\quad & \max\ c_i - \boldsymbol{x}^{0^{\mathrm{T}}}\boldsymbol{c} \\
& \text{subject to} \\
& \boldsymbol{w}\boldsymbol{d}_k^*(h)^{\mathrm{T}}\boldsymbol{c} \le -w d_{0k}^*(h),\ k = 1,2,\dots,v\,.
\end{aligned}
$$

Let $\hat{\boldsymbol{c}}_i$ be an obtained optimal solution to the i-th problem P(4.33).

Step 3: For $i = 1,2,\dots,n$, if $\hat{\boldsymbol{c}}_i^{\mathrm{T}}\boldsymbol{x}^0 > q^0$ then update $l_i = l_i + 1$ and let $\boldsymbol{c}_{il} = \hat{\boldsymbol{c}}_i$. If no l_i is updated, we terminate the algorithm and obtain an optimal solution $\boldsymbol{x}^0$ to problem P(4.32).

Step 4: Solve a linear programming problem,

$$
\begin{aligned}
\mathbf{P(4.34)}\quad & \min\ q \\
& \text{subject to} \\
& c_i - \boldsymbol{c}_{iw}^{\mathrm{T}}\boldsymbol{x} \le q,\ w = 1,2,\dots,l_i,\ i = 1,2,\dots,n\,, \\
& \boldsymbol{e}^{\mathrm{T}}\boldsymbol{x} = 1\,, \\
& \boldsymbol{x} \ge \boldsymbol{0}\,.
\end{aligned}
$$

Let $(\boldsymbol{x}^{0^{\mathrm{T}}}, q^0)^{\mathrm{T}}$ be an obtained optimal solution. Return to Step 2.

In this algorithm, we solve two kinds of linear programming problems; therefore, we can also solve problem P(4.32) using linear programming techniques only.

4.3.4 Numerical Illustration

In this section, we give examples of necessity fractile optimization models and minimax regret models with scenario decomposed fuzzy numbers and an oblique fuzzy vector because all of them are reduced to linear programming problems.

Example 4.4. *As an example of scenario decomposed fuzzy numbers, we consider a case where the possible range of return rates of five assets in different categories of industry estimated by the following fuzzy if-then rules:*

- *if s is small then the return rate vector γ is in a possible range $\boldsymbol{C}_1$,*
- *if s is medium then the return rate vector γ is in a possible range $\boldsymbol{C}_2$,*
- *if s is large then the return rate vector γ is in a possible range $\boldsymbol{C}_3$,*

where fuzzy sets small, medium and large are triangular fuzzy numbers depicted in Fig. 4.10. $\boldsymbol{C}_i$, $i = 1,2,3$ are non-interactive fuzzy numbers whose component C_{ij} has the following type of membership function:

$$\mu_{C_{ij}}(r) = \exp\left(-\frac{(r - c^{c}_{ij})^2}{w_{ij}}\right).$$

Parameters c^{c}_{ij} and w_{ij}, $i = 1,2,3$, $j = 1,2,\ldots,5$ are defined by the values in Table 4.5. The estimated possible range S of scenario variable is a triangular fuzzy number depicted in Fig. 4.10.

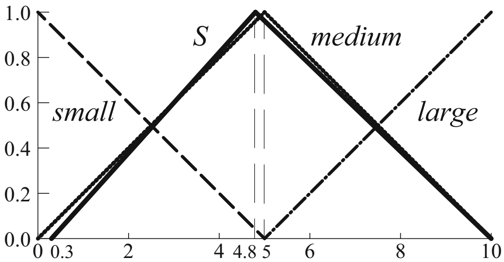


Fig. 4.10 Fuzzy sets *small*, *medium* and *large* and fuzzy set S

Table 4.5 Parameters of C_{ij}

s	i	c_{i1}	w_{i1}	c_{i2}	w_{i2}	c_{i3}	w_{i3}	c_{i4}	w_{i4}	c_{i5}	w_{i5}
Small	1	0.18	0.023	0.22	0.05	0.26	0.01	0.2	0.026	0.2	0.008
Medium	2	0.25	0.023	0.22	0.014	0.19	0.013	0.16	0.005	0.14	0.004
Large	3	0.3	0.03	0.18	0.015	0.18	0.0225	0.2	0.006	0.13	0.004

Let $\alpha = 0.7$. We obtain $\bar{s}^{\mathrm{L}} = 3.45$ and $\bar{s}^{\mathrm{R}} = 6.36$. The necessity fractile optimization model is formulated as

P(4.35) max z

subject to

$$\begin{aligned}
&0.08359x_1 + 0.09017x_2 + 0.06489x_3 + 0.08241x_4 + 0.07060x_5 \geq z,\\
&0.06189x_1 + 0.05436x_2 + 0.09136x_3 + 0.06402x_4 + 0.08029x_5 \geq z,\\
&0.09076x_1 + 0.07805x_2 + 0.05143x_3 + 0.09128x_4 + 0.06788x_5 \geq z,\\
&x_1 + x_2 + x_3 + x_4 + x_5 = 1,\\
&x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Solving this problem, we obtain the optimal solution shown in Table 4.6.

On the other hand, the minimum regret model with $F(r_1, r_2) = r_1 - r_2$ and $\alpha = 0.7$ is formulated as

P(4.36) min q

subject to

$$\begin{aligned}
&0.41641x_1 + 0.09017x_2 + 0.06489x_3 + 0.08241x_4 + 0.07060x_5 + q \geq 0.41641,\\
&0.08359x_1 + 0.34983x_2 + 0.06489x_3 + 0.08241x_4 + 0.07060x_5 + q \geq 0.34983,\\
&0.08359x_1 + 0.09017x_2 + 0.31511x_3 + 0.08241x_4 + 0.07060x_5 + q \geq 0.31511,\\
&0.08359x_1 + 0.09017x_2 + 0.06489x_3 + 0.23759x_4 + 0.07060x_5 + q \geq 0.23759,\\
&0.08359x_1 + 0.09017x_2 + 0.06489x_3 + 0.08241x_4 + 0.20940x_5 + q \geq 0.20940,\\
&0.39471x_1 + 0.05436x_2 + 0.09136x_3 + 0.06402x_4 + 0.08029x_5 + q \geq 0.39471,\\
&0.06189x_1 + 0.38564x_2 + 0.09136x_3 + 0.06402x_4 + 0.08029x_5 + q \geq 0.38564,\\
&0.06189x_1 + 0.05436x_2 + 0.33204x_3 + 0.06402x_4 + 0.08029x_5 + q \geq 0.33204,\\
&0.06189x_1 + 0.05436x_2 + 0.09136x_3 + 0.28078x_4 + 0.08029x_5 + q \geq 0.28078,\\
&0.06189x_1 + 0.05436x_2 + 0.09136x_3 + 0.06402x_4 + 0.23691x_5 + q \geq 0.23691,\\
&0.43644x_1 + 0.07805x_2 + 0.05143x_3 + 0.09128x_4 + 0.06788x_5 + q \geq 0.43644,\\
&0.09076x_1 + 0.34019x_2 + 0.05143x_3 + 0.09128x_4 + 0.06788x_5 + q \geq 0.34019,\\
&0.09076x_1 + 0.07805x_2 + 0.32313x_3 + 0.09128x_4 + 0.06788x_5 + q \geq 0.32313,\\
&0.09076x_1 + 0.07805x_2 + 0.05143x_3 + 0.25048x_4 + 0.06788x_5 + q \geq 0.25048,\\
&0.09076x_1 + 0.07805x_2 + 0.05143x_3 + 0.09128x_4 + 0.20668x_5 + q \geq 0.20668,\\
&x_1 + x_2 + x_3 + x_4 + x_5 = 1,\\
&x_1, x_2, x_3, x_4, x_5 \geq 0.
\end{aligned}$$

Solving this problem, we obtain the optimal solution shown in Table 4.6.

Table 4.6 Solutions of problems P(4.35) and P(4.36)

Model	x_1	x_2	x_3	x_4	x_5	z/q
Necessity fractile (P(4.35))	0	0	0.33456	0.54488	0.12056	0.07513
Minimax regret (P(4.36))	0.42766	0.32750	0.22807	0.01677	0	0.210966

Comparing the solutions in Table 4.6, we observe that the solution of the necessity fractile optimization model suggests the investment to asset whose estimated return rates are small and their variations are small while the solution to the minimax regret model suggests the investment to asset whose estimated rate of return are large and their variations are large. In the necessity fractile optimization model, only the minimal return rates of assets are used to estimate the worst case and thus a pessimistic solution is obtained. On the other hand, in the minimax regret model, the maximal return rates of assets are also used to estimate the worst regret and thus the solution is not very pessimistic. Finally, note that the solutions are easily changed by a small change of parameters because the five assets are comparable.

Example 4.5. *As an example of the oblique fuzzy vector, we consider a case where D is given by*

$$D = \begin{pmatrix} 1 & 7 & -1.5 & 1 & -6 \\ 0 & 20 & 20 & 10 & 3 \\ 0 & 0 & 0.5 & 3 & 3 \\ 0 & 0 & 0 & 3 & 2 \\ 4 & 0 & 6 & -6.5 & 3 \end{pmatrix},$$

and B_i, $i = 1, 2, \ldots, 5$ is defined by membership function,

$$\mu_{B_i}(r) = \exp\left(-\frac{(r - b_i^c)^2}{s_i}\right),$$

with the following parameters:

$$b_1^c = 0.396,\ b_2^c = 11.194,\ b_3^c = 1.396,\ b_4^c = 1.078,\ b_5^c = 1.463,$$
$$s_1 = 0.008,\ s_2 = 0.0025,\ s_3 = 0.0036,\ s_4 = 0.0009,\ s_5 = 0.006.$$

In this case, we have

$$D^{-1} = \begin{pmatrix} 0.091265 & -0.031943 & -1.174707 & 1.742992 & 0.227184 \\ 0.170795 & -0.009778 & 1.415906 & -1.532757 & -0.042699 \\ -0.156454 & 0.054759 & -1.129074 & 1.083442 & 0.039113 \\ -0.052151 & 0.018253 & -1.043025 & 1.361147 & 0.013038 \\ 0.078227 & -0.027379 & 1.564537 & -1.541721 & -0.019557 \end{pmatrix}$$

and the membership function of marginal fuzzy set C_i, $i = 1, 2, \ldots, 5$ is obtained using equation (4.25) with parameters [60]:

$$c_1^c = 0.25,\ c_2^c = 0.22,\ c_3^c = 0.2,\ c_4^c = 0.214,\ c_5^c = 0.218,$$
$$w_1 = 0.022539,\ w_2 = 0.022503,\ w_3 = 0.014402,\ w_4 = 0.012101,\ w_5 = 0.022501.$$

Applying the necessity fractile model with $\alpha = 0.7$, we solve the following linear programming problem:

$$\begin{aligned}
\max\ & 0.297858y_1^+ + 11.139137y_2^+ + 1.330165y_3^+ + 1.045082y_4^+ \\
& +1.378007y_5^+ - 0.494142y_1^- - 11.248863y_2^- \\
& -1.461835y_3^- - 1.110918y_4^- - 1.547993y_5^- \\
\text{subject to}\ & \\
& y_1^+ + 4y_5^+ - y_1^- - 4y_5^- \geq 0, \\
& 7y_1^+ + 20y_2^+ - 7y_1^- - 20y_2^- \geq 0, \\
& -1.5y_1^+ + 20y_2^+ + 0.5y_3^+ + 6y_5^+ \\
& \qquad +1.5y_1^- - 20y_2^- - 0.5y_3^- - 6y_5^- \geq 0, \\
& y_1^+ + 10y_2^+ + 3y_3^+ + 3y_4^+ - 6.5y_5^+ \\
& \qquad -y_1^- - 10y_2^- - 3y_3^- - 3y_4^- + 6.5y_5^- \geq 0, \\
& -6y_1^+ + 3y_2^+ + 3y_3^+ + 2y_4^+ + 3y_5^+ \\
& \qquad +6y_1^- - 3y_2^- - 3y_3^- - 2y_4^- - 3y_5^- \geq 0, \\
& 1.5y_1^+ + 53y_2^+ + 6.5y_3^+ + 5y_4^+ + 6.5y_5^+ \\
& \qquad -1.5y_1^- - 53y_2^- - 6.5y_3^- - 5y_4^- - 6.5y_5^- = 1, \\
& y_i^+ \geq 0,\ y_i^- \geq 0,\ i = 1, 2, \ldots, 5.
\end{aligned}$$

We obtain an optimal solution of the above problem as

$$\begin{aligned}
& y_1^+ = 0,\ y_2^+ = 0.015873,\ y_3^+ = 0,\ y_4^+ = 0,\ y_5^+ = 0.024420, \\
& y_1^- = 0,\ y_2^- = 0,\ y_3^- = 0, y_4^- = 0, y_5^- = 0.
\end{aligned}$$

Then the optimal investment rate proportions $\boldsymbol{x}$ is obtained by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = D^{\mathrm{T}} \left(\begin{pmatrix} y_1^+ \\ y_2^+ \\ y_3^+ \\ y_4^+ \\ y_5^+ \end{pmatrix} - \begin{pmatrix} y_1^- \\ y_2^- \\ y_3^- \\ y_4^- \\ y_5^- \end{pmatrix} \right) = \begin{pmatrix} 0.097680 \\ 0.317460 \\ 0.463981 \\ 0 \\ 0.120879 \end{pmatrix}.$$

Note that in this example, we use problem P(4.28) and solve it by a linear programming technique because the problem size is not very big. However when the problem size is rather big, we can use problem P(4.27) and utilize Bender's decomposition method so that we can obtain an optimal solution by solving smaller linear programming problems [60].

On the other hand, applying minimax regret model with $F(r_1, r_2) = r_1 - r_2$ and $\alpha = 0.7$, we formulated problem P(4.29). The problem is written as

$$\begin{aligned}
\min\ & q \\
\text{subject to}\ & \\
& 0.297858y_{11}^+ + 11.139137y_{12}^+ + 1.330165y_{13}^+ + 1.045082y_{14}^+ + 1.378007y_{15}^+ \\
& -0.494142y_{11}^- - 11.248863y_{12}^- - 1.461835y_{13}^- - 1.110918y_{14}^- - 1.547993y_{15}^- \geq q, \\
& x_1 + y_{11}^+ + 4y_{15}^+ - y_{11}^- - 4y_{15}^- = 1,
\end{aligned}$$

$$
\begin{aligned}
&-x_2 + 7y_{11}^+ + 20y_{12}^+ - 7y_{11}^- - 20y_{12}^- = 0,\\
&-x_3 - 1.5y_{11}^+ + 20y_{12}^+ + 0.5y_{13}^+ + 6y_{15}^+ + 1.5y_{11}^- - 20y_{12}^- - 0.5y_{13}^- - 6y_{15}^- = 0,\\
&-x_4 + y_{11}^+ + 10y_{12}^+ + 3y_{13}^+ + 3y_{14}^+ - 6.5y_{15}^+ - y_{11}^- - 10y_{12}^- - 3y_{13}^- - 3y_{14}^- + 6.5y_{15}^- = 0,\\
&-x_5 - 6y_{11}^+ + 3y_{12}^+ + 3y_{13}^+ + 2y_{14}^+ + 3y_{15}^+ + 6y_{11}^- - 3y_{12}^- - 3y_{13}^- - 2y_{14}^- - 3y_{15}^- = 0,\\
&0.297858y_{21}^+ + 11.139137y_{22}^+ + 1.330165y_{23}^+ + 1.045082y_{24}^+ + 1.378007y_{25}^+\\
&-0.494142y_{21}^- - 11.248863y_{22}^- - 1.461835y_{23}^- - 1.110918y_{24}^- - 1.547993y_{25}^- \geq q,\\
&-x_1 + y_{21}^+ + 4y_{25}^+ - y_{21}^- - 4y_{25}^- = 0,\\
&x_2 + 7y_{21}^+ + 20y_{22}^+ - 7y_{21}^- - 20y_{22}^- = 1,\\
&-x_3 - 1.5y_{21}^+ + 20y_{22}^+ + 0.5y_{23}^+ + 6y_{25}^+ + 1.5y_{21}^- - 20y_{22}^- - 0.5y_{23}^- - 6y_{25}^- = 0,\\
&-x_4 + y_{21}^+ + 10y_{22}^+ + 3y_{23}^+ + 3y_{24}^+ - 6.5y_{25}^+ - y_{21}^- - 10y_{22}^- - 3y_{23}^- - 3y_{24}^- + 6.5y_{25}^- = 0,\\
&-x_5 - 6y_{21}^+ + 3y_{22}^+ + 3y_{23}^+ + 2y_{24}^+ + 3y_{25}^+ + 6y_{21}^- - 3y_{22}^- - 3y_{23}^- - 2y_{24}^- - 3y_{25}^- = 0,\\
&0.297858y_{31}^+ + 11.139137y_{32}^+ + 1.330165y_{33}^+ + 1.045082y_{34}^+ + 1.378007y_{35}^+\\
&-0.494142y_{31}^- - 11.248863y_{32}^- - 1.461835y_{33}^- - 1.110918y_{34}^- - 1.547993y_{35}^- \geq q,\\
&-x_1 + y_{31}^+ + 4y_{35}^+ - y_{31}^- - 4y_{35}^- = 0,\\
&-x_2 + 7y_{31}^+ + 20y_{32}^+ - 7y_{31}^- - 20y_{32}^- = 0,\\
&x_3 - 1.5y_{31}^+ + 20y_{32}^+ + 0.5y_{33}^+ + 6y_{35}^+ + 1.5y_{31}^- - 20y_{32}^- - 0.5y_{33}^- - 6y_{35}^- = 1,\\
&-x_4 + y_{31}^+ + 10y_{32}^+ + 3y_{33}^+ + 3y_{34}^+ - 6.5y_{35}^+ - y_{31}^- - 10y_{32}^- - 3y_{33}^- - 3y_{34}^- + 6.5y_{35}^- = 0,\\
&-x_5 - 6y_{31}^+ + 3y_{32}^+ + 3y_{33}^+ + 2y_{34}^+ + 3y_{35}^+ + 6y_{31}^- - 3y_{32}^- - 3y_{33}^- - 2y_{34}^- - 3y_{35}^- = 0,\\
&0.297858y_{41}^+ + 11.139137y_{42}^+ + 1.330165y_{43}^+ + 1.045082y_{44}^+ + 1.378007y_{45}^+\\
&-0.494142y_{41}^- - 11.248863y_{42}^- - 1.461835y_{43}^- - 1.110918y_{44}^- - 1.547993y_{45}^- \geq q,\\
&-x_1 + y_{41}^+ + 4y_{45}^+ - y_{41}^- - 4y_{45}^- = 0,\\
&-x_2 + 7y_{41}^+ + 20y_{42}^+ - 7y_{41}^- - 20y_{42}^- = 0,\\
&-x_3 - 1.5y_{41}^+ + 20y_{42}^+ + 0.5y_{43}^+ + 6y_{45}^+ + 1.5y_{41}^- - 20y_{42}^- - 0.5y_{43}^- - 6y_{45}^- = 0,\\
&x_4 + y_{41}^+ + 10y_{42}^+ + 3y_{43}^+ + 3y_{44}^+ - 6.5y_{45}^+ - y_{41}^- - 10y_{42}^- - 3y_{43}^- - 3y_{44}^- + 6.5y_{45}^- = 1,\\
&-x_5 - 6y_{41}^+ + 3y_{42}^+ + 3y_{43}^+ + 2y_{44}^+ + 3y_{45}^+ + 6y_{41}^- - 3y_{42}^- - 3y_{43}^- - 2y_{44}^- - 3y_{45}^- = 0,\\
&0.297858y_{51}^+ + 11.139137y_{52}^+ + 1.330165y_{53}^+ + 1.045082y_{54}^+ + 1.378007y_{55}^+\\
&-0.494142y_{51}^- - 11.248863y_{52}^- - 1.461835y_{53}^- - 1.110918y_{54}^- - 1.547993y_{55}^- \geq q,\\
&-x_1 + y_{51}^+ + 4y_{55}^+ - y_{51}^- - 4y_{55}^- = 0,\\
&-x_2 + 7y_{51}^+ + 20y_{52}^+ - 7y_{51}^- - 20y_{52}^- = 0,\\
&-x_3 - 1.5y_{51}^+ + 20y_{52}^+ + 0.5y_{53}^+ + 6y_{55}^+ + 1.5y_{51}^- - 20y_{52}^- - 0.5y_{53}^- - 6y_{55}^- = 0,\\
&-x_4 + y_{51}^+ + 10y_{52}^+ + 3y_{53}^+ + 3y_{54}^+ - 6.5y_{55}^+ - y_{51}^- - 10y_{52}^- - 3y_{53}^- - 3y_{54}^- + 6.5y_{55}^- = 0,\\
&x_5 - 6y_{51}^+ + 3y_{52}^+ + 3y_{53}^+ + 2y_{54}^+ + 3y_{55}^+ + 6y_{51}^- - 3y_{52}^- - 3y_{53}^- - 2y_{54}^- - 3y_{55}^- = 1,\\
&x_1 + x_2 + x_3 + x_4 + x_5 = 1,\\
&x_i \geq 0,\ y_{ij}^+ \geq 0,\ y_{ij}^- \geq 0,\ i = 1, 2, \ldots, 5,\ j = 1, 2, \ldots, 5.
\end{aligned}
$$

Solving this problem, we obtain an optimal solution q, $\boldsymbol{x} = (x_1, x_2, x_3, x_4, x_5)^{\mathrm{T}}$, $Y^+ = (y_{ij}^+)$ and $Y^- = (y_{ij}^-)$ as follows:

$$
q = 0.440294,\quad \boldsymbol{x} = (0.159177,\ 0.328070,\ 0.043885,\ 0.061670,\ 0.407198)^{\mathrm{T}},
$$

$$
Y^+ = \begin{pmatrix}
0.154542 & 0 & 0 & 0.466399 & 0.171570\\
0.151061 & 0 & 1.287608 & 0 & 0.002029\\
0 & 0.034040 & 0 & 0.266643 & 0.052392\\
0.046613 & 0 & 0 & 0.471559 & 0.028141\\
0.106851 & 0 & 1.091117 & 0 & 0.013082
\end{pmatrix},
$$

$$
Y^- = \begin{pmatrix}
0 & 0.037686 & 0 & 0 & 0\\
0 & 0.019275 & 0 & 1.248760 & 0\\
0.050390 & 0 & 0.229241 & 0 & 0\\
0 & 0 & 0.113644 & 0 & 0\\
0 & 0.020994 & 0 & 1.007853 & 0
\end{pmatrix}.
$$

The above problem can be solved by the Bender's decomposition method so that the problem is decomposed into five subproblems and one master problem [60].

We observe the difference between those solutions: the solution of the necessity fractile optimization model suggests large amounts of investment to x_2 and x_3 while the solution of the minimax regret model suggests large amounts of investment to x_2 and x_5.

Finally to see the significance of the interaction, we solve the problem with non-interactive fuzzy numbers having the same marginal membership functions. Namely, this problem can be seen as the problem discarding the interaction. We obtain

$$\boldsymbol{x} = (0, 0, 0, 1, 0)^{\mathrm{T}}$$

from necessity fractile optimization model, and

$$q = 0.4640220, \ \boldsymbol{x} = (0.255508, 0.190326, 0.179810, 0.155279, 0.219077)^{\mathrm{T}}$$

from the conventional minimax regret model. We observe the big differences of solutions between problems with oblique fuzzy vector and with non-interactive fuzzy numbers. In necessity fractile optimization model, while the solution to the problem discarding the interaction suggests the concentrate investment on the fourth asset, the solution to the problem taking care of the interaction suggests no investment on the fourth asset. Similarly, in the conventional minimax regret model, while the solution to the problem discarding the interaction suggests the investment of more than 15% of the fund on the fourth asset, the solution to the problem taking care of the interaction suggests the investment of less than 7% of the fund. These facts show that the fourth asset is not very attractive one if we consider the interaction among the return rates of five assets. Moreover, the distributions of the fund are significantly different between the solutions of the two conventional minimax regret models. By these results, we understand the significance of the interaction.

4.4 Comments

In this chapter, we have presented the following facts:

- The classical possibilistic programming approaches to simple portfolio selection problems with fuzzy coefficients have been discussed.
- When fuzzy coefficients are non-interactive, the reduced problems obtained by the classical possibilistic programming approach are tractable. They are linear programming problems. However, the obtained solutions indicate concentrated investments.

- To obtain diversified investment solutions, minimax regret approach has been presented. It is shown that the reduced problem becomes a linear programming problem.
- Moreover, three models to treat interaction among fuzzy coefficients without loss of the tractability of the reduced problem are described. Using those models together with minimax regret approach, more diversified investment solutions can be obtained.

Chapter 5
Portfolio Optimization Using Credibility Theory

Abstract. In this chapter, we present a hybrid bi-objective credibility-based fuzzy mathematical programming model for portfolio selection under fuzzy environment. The expected value and chance constrained programming techniques are used to formulate the mathematical model in which return, risk and liquidity are considered for measuring performance of an asset. The model seeks to maximize the portfolio return while minimizing the portfolio risk. The portfolio liquidity is considered as a constraint. To solve the fuzzy optimization model, a two-phase approach is discussed.

5.1 Credibility Theory

The possibility measure widely used in literature to deal with fuzzy variables that represent return rates of the assets in portfolio theory does not obey the law of truth conservation. Further, it is inconsistent with the law of excluded middle and the law of contradiction. For example, a fuzzy event may fail even though its possibility value is 1 and hold even though its necessity value is 0. This is mainly due to the fact that possibility measure does not satisfy self-duality property which is absolutely needed in both theory and practice. In order to elevate this difficulty, Liu and Liu [85] presented a self-dual measure, namely, credibility measure. Note that when the credibility value of a fuzzy event attains 1, the fuzzy event will surely happen; however, when the corresponding possibility value achieves 1, the fuzzy event may fail to happen. In other words, the fuzzy event must hold if its credibility value is 1 and fail if its credibility value is 0. Credibility theory, founded by Liu [87] in 2004 and refined by Liu [88] in 2007, is a branch of mathematics for studying the behavior of fuzzy phenomena. Mathematically, it can be described as follows.

Let Θ be a nonempty set (representing the sample space) and $\mathcal{P}(\Theta)$ be the power set of Θ (i.e., all possible subsets of Θ). Each element in $\mathcal{P}(\Theta)$ is called an event. To present an axiomatic definition of credibility, it is necessary

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to assign to each event A, a number $Cr\{A\}$, which represents the credibility that A will occur. Further, to ensure that the number $Cr\{A\}$ has certain mathematical properties which we intuitively expect credibility to have, the following four axioms must hold:

Axiom 1. *(Normality)* $Cr\{\Theta\} = 1$.
Axiom 2. *(Monotonicity)* $Cr\{A\} \leq Cr\{B\}$ *whenever* $A \subset B$.
Axiom 3. *(Self-Duality)* $Cr\{A\} + Cr\{A^c\} = 1$ *for any event* $A \in \mathcal{P}(\Theta)$.
Axiom 4. *(Maximality)* $Cr\{\cup_i A_i\} \wedge 0.5 = \sup_i Cr\{A_i\}$ *for any events* $\{A_i\}$ *with* $Cr\{A_i\} \leq 0.5$.

The first three axioms are self explanatory. The maximum axiom may be understood as follows. There is no uncertainty in the outcome of an event if its credibility measure is 1 (or 0) because we may believe that the event occurs (or not). On the other hand, an event is the most uncertain if its credibility measure is 0.5 since in such a case both the event and its complement may be regarded as 'equally likely'. Further, if there is no information about the credibility measure of an event then we should consider it as 0.5. Based on this argument, Liu [86] proposed the maximum uncertainty principle which states that 'For any event, if there are multiple reasonable values that a credibility measure may take, then the value as close to 0.5 as possible is assigned to it'.

Definition 5.1 (Credibility measure). The set function Cr is called a *credibility measure* if it satisfies the normality, monotonicity, self-duality, and maximality axioms.

Example 5.1. *Let* $\Theta = \{\theta_1, \theta_2\}$. *There are only four possible events for this case:* $A_1 = \phi$, $A_2 = \{\theta_1\}$, $A_3 = \{\theta_2\}$ $A_4 = \Theta = \{\theta_1, \theta_2\}$. *Define* $Cr\{A_1\} = 0$, $Cr\{A_2\} = 0.7$, $Cr\{A_3\} = 0.3$, *and* $Cr\{A_4\} = 1$. *To verify wether the set function* Cr *is a credibility measure or not, we need to check the four axioms.*
Axiom 1. *The set function* Cr *satisfies the normality as* $Cr\{\Theta\} = 1$.
Axiom 2. *The set function* Cr *satisfies the monotonicity, for example,*

$$A_1 \subset A_2 \text{ and } Cr\{A_1\} < Cr\{A_2\}$$
$$A_1 \subset A_3 \text{ and } Cr\{A_1\} < Cr\{A_3\}$$
$$A_1 \subset A_4 \text{ and } Cr\{A_1\} < Cr\{A_4\}$$
$$A_2 \subset A_4 \text{ and } Cr\{A_2\} < Cr\{A_4\}$$
$$A_3 \subset A_4 \text{ and } Cr\{A_3\} < Cr\{A_4\}$$

Axiom 3. *The set function* Cr *satisfies the self-duality, for example,*

$$Cr\{A_1\} + Cr\{A_1^c\}(= Cr\{A_4\}) = 1$$

Axiom 4. *The set function* Cr *satisfies the maximality, for example,*

$$Cr\{A_1 \cup A_3\}(= Cr\{A_3\}) \wedge 0.5 = \sup\{Cr\{A_1\}, Cr\{A_3\}\}$$

Hence, the set function Cr *is a credibility measure.*

The following theorems present additional properties of credibility measure and are established in [85, 88].

Theorem 5.1. *Let* Θ *be a nonempty set,* $\mathcal{P}(\Theta)$ *the power set of* Θ *and* Cr *the credibility measure. Then* $Cr\{\phi\} = 0$ *and* $0 \leq Cr\{A\} \leq 1$ *for any* $A \in \mathcal{P}$.

Proof. From Axioms 1 and 3, we have $Cr\{\phi\} = 1 - Cr\{\Theta\} = 1 - 1 = 0$. Also, $\phi \subset A \subset \Theta$, we have $Cr\{\phi\} \le Cr\{A\} \le Cr\{\Theta\}$ from Axiom 2, i.e., $0 \le Cr\{A\} \le 1$. □

Theorem 5.2. *The credibility measure is subadditive. That is,*

$$Cr\{A \cup B\} \le Cr\{A\} + Cr\{B\}$$

for any events A and B. Further, the credibility measure is null-additive, i.e., $Cr\{A \cup B\} = Cr\{A\} + Cr\{B\}$ if either $Cr\{A\} = 0$ or $Cr\{B\} = 0$.

Proof. In order to prove the theorem, we need to consider the following three cases.

Case 1: $Cr\{A\} < 0.5$ and $Cr\{B\} < 0.5$. Using Axiom 4, we have

$$Cr\{A \cup B\} = Cr\{A\} \vee Cr\{B\} \le Cr\{A\} + Cr\{B\}.$$

Case 2: $Cr\{A\} \ge 0.5$. Using Axioms 2 and 3, we have $Cr\{A^c\} \le 0.5$ and $Cr\{A \cup B\} \ge Cr\{A\} \ge 0.5$. Then

$$\begin{aligned} Cr\{A^c\} &= Cr\{A^c \cap B\} \vee Cr\{A^c \cap B^c\} \\ &\le Cr\{A^c \cap B\} + Cr\{A^c \cap B^c\} \\ &\le Cr\{B\} + Cr\{A^c \cap B^c\}. \end{aligned}$$

Using the above inequality, we have

$$\begin{aligned} Cr\{A\} + Cr\{B\} &= 1 - Cr\{A^c\} + Cr\{B\} \\ &\ge 1 - Cr\{B\} - Cr\{A^c \cap B^c\} + Cr\{B\} \\ &= 1 - Cr\{A^c \cap B^c\} \\ &= Cr\{A \cup B\}. \end{aligned}$$

Case 3: $Cr\{B\} \ge 0.5$. Using Axioms 2 and 3, we have $Cr\{B^c\} \le 0.5$ and $Cr\{A \cup B\} \ge Cr\{B\} \ge 0.5$. Then

$$\begin{aligned} Cr\{B^c\} &= Cr\{A \cap B^c\} \vee Cr\{A^c \cap B^c\} \\ &\le Cr\{A \cap B^c\} + Cr\{A^c \cap B^c\} \\ &\le Cr\{A\} + Cr\{A^c \cap B^c\}. \end{aligned}$$

Using the above inequality, we have

$$\begin{aligned} Cr\{A\} + Cr\{B\} &= Cr\{A\} + 1 - Cr\{B^c\} \\ &\ge 1 - Cr\{A\} - Cr\{A^c \cap B^c\} + Cr\{A\} \\ &= 1 - Cr\{A^c \cap B^c\} \\ &= Cr\{A \cup B\}. \end{aligned}$$

Thus, the subadditivity property is established. Further, it follows from subadditivity property that $Cr\{A \cup B\} = Cr\{A\} + Cr\{B\}$ if either $Cr\{A\} = 0$ or $Cr\{B\} = 0$. □

Definition 5.2 (Credibility space). Let Θ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of Θ and Cr the credibility measure. Then the triplet $(\Theta, \mathcal{P}(\Theta), Cr)$ is called a credibility space.

Definition 5.3 (Fuzzy variable). A fuzzy variable is a (measurable) function from a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ to the set of real numbers.

Example 5.2. *Let* $\Theta = \{\theta_1, \theta_2\}$ *with* $Cr\{\theta_1\} = Cr\{\theta_2\} = 0.5$. *Then the function*

$$\xi(\theta) = \begin{cases} 0, \text{ if } \theta = \theta_1, \\ 1, \text{ if } \theta = \theta_2, \end{cases}$$

define a fuzzy variable.

Remark 5.1. *Since a fuzzy variable* ξ *is a function on a credibility space, for any set* A *of real numbers, the set*

$$\{\xi \in A\} = \{\theta \in \Theta | \xi(\theta) \in A\}$$

is always an element in $\mathcal{P}$. *In other words, the fuzzy variable* ξ *is always a measurable function and* $\{\xi \in A\}$ *is always an event.*

Note that if ξ_1 and ξ_2 are two fuzzy variables defined on a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$, then $\xi_1 = \xi_2$ implies $\xi_1(\theta) = \xi_2(\theta)$ for almost all $\theta \in \Theta$.

Definition 5.4 (Membership function). Let ξ be a fuzzy variable defined on the credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$. Then its membership function is derived from credibility measure using the following relation

$$\mu(r) = (2Cr\{\xi = r\}) \wedge 1, \quad r \in \mathbf{R}.$$

The membership function represents the degree that the fuzzy variable ξ takes some prescribed value. The membership degree $\mu(r) = 0$ if r is an impossible point and $\mu(r) = 1$ if r is the most possible point that ξ takes. Note that a fuzzy variable has a unique membership function; however, a membership function may produce multiple fuzzy variables.

Example 5.3. *Let* $\Theta = \{\theta_1, \theta_2\}$ *with* $Cr\{\theta_1\} = Cr\{\theta_2\} = 0.5$. *Define the functions*

$$\xi_1(\theta) = \begin{cases} 0, \text{ if } \theta = \theta_1, \\ 1, \text{ if } \theta = \theta_2, \end{cases} \quad \xi_2(\theta) = \begin{cases} 1, \text{ if } \theta = \theta_1, \\ 0, \text{ if } \theta = \theta_2. \end{cases}$$

It is clear that both of them are fuzzy variables and have the same membership function, $\mu(r) \equiv 1$ *on* $r = 0$ *or* 1.

An important issue that needs to be addressed now is that *if the membership function of a fuzzy variable* ξ *is known to us, then how can we determine the credibility value of a fuzzy event?* To answer this question, we have to rely on the following credibility inversion theorem proved by Liu and Liu [85].

Theorem 5.3. *Let ξ be a fuzzy variable with membership function μ. Then for any set A of real numbers, we have*

$$Cr\{\xi \in A\} = \frac{1}{2}\left(\sup_{r\in A}\mu(r) + 1 - \sup_{r\in A^c}\mu(r)\right). \tag{5.1}$$

Proof. In order to prove the theorem, we consider two cases:
Case 1: If $Cr\{\xi \in A\} \leq 0.5$, then using Axiom 2, we have $Cr\{\xi = r\} \leq 0.5$ for each $r \in A$. From Axiom 4 we have

$$Cr\{\xi \in A\} = \frac{1}{2}\left(\sup_{r\in A}(2Cr\{\xi = r\} \wedge 1)\right) = \frac{1}{2}\sup_{r\in A}\mu(r). \tag{5.2}$$

Also, using Axiom 3, we have $Cr\{\xi \in A^c\} \geq 0.5$ and $\sup_{r\in A^c} Cr\{\xi = r\} \geq 0.5$. Therefore, we have

$$\sup_{r\in A^c}\mu(r) = \sup_{r\in A^c}(2Cr\{\xi = r\} \wedge 1) = 1. \tag{5.3}$$

From (5.2) and (5.3) it is clear that (5.1) holds.
Case 2: If $Cr\{\xi \in A\} \geq 0.5$, then $Cr\{\xi \in A^c\} \leq 0.5$. Now from case 1 it follows that

$$Cr\{\xi \in A\} = 1 - Cr\{\xi \in A^c\} = 1 - \frac{1}{2}\left(\sup_{r\in A^c}\mu(r) + 1 - \sup_{r\in A}\mu(r)\right)$$
$$= \frac{1}{2}\left(\sup_{r\in A}\mu(r) + 1 - \sup_{r\in A^c}\mu(r)\right).$$

□

Remark 5.2. *Let ξ be a fuzzy variable with membership function μ. Then the following equations can follow from Theorem 5.3.*

$$Cr\{\xi = r\} = \frac{1}{2}\left(\mu(r) + 1 - \sup_{y\neq r}\mu(y)\right), \quad \forall r \in \mathbf{R};$$
$$Cr\{\xi \leq r\} = \frac{1}{2}\left(\sup_{y\leq r}\mu(y) + 1 - \sup_{y>r}\mu(y)\right), \quad \forall r \in \mathbf{R};$$
$$Cr\{\xi \geq r\} = \frac{1}{2}\left(\sup_{y\geq r}\mu(y) + 1 - \sup_{y<r}\mu(y)\right), \quad \forall r \in \mathbf{R}.$$

Further, if μ is a continuous function, then

$$Cr\{\xi = r\} = \frac{\mu(r)}{2}, \quad \forall r \in \mathbf{R}.$$

Credibility of Some Special Fuzzy Variables

Definition 5.5 (Triangular fuzzy variable). A fuzzy variable is said to be a *triangular fuzzy variable* if it is fully determined by the triplet (a, b, c) of crisp numbers with $a < b < c$ and its membership function (see Fig. 5.1) is given by

$$\mu(r) = \begin{cases} \dfrac{r-a}{b-a}, & \text{if } a \le r \le b\,, \\ \dfrac{c-r}{c-b}, & \text{if } b \le r \le c\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

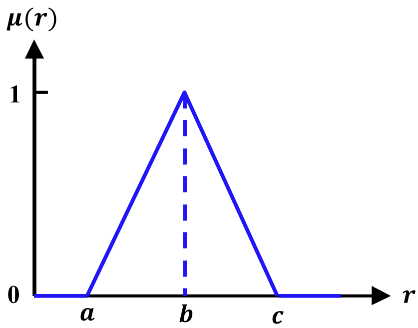

Fig. 5.1 Membership function of triangular fuzzy variable

To find the credibility $Cr\{\xi \le r\}$, we use the credibility inversion theorem. Accordingly, if $c \le r$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y \le r} \mu(y) + 1 - \sup_{y > r} \mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}(1 + 1 - 0) = 1\,.$$

If $b \le r \le c$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y \le r} \mu(y) + 1 - \sup_{y > r} \mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}\left(1 + 1 - \frac{c-r}{c-b}\right) = \frac{c - 2b + r}{2(c-b)}\,.$$

If $a \le r \le b$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y \le r} \mu(y) + 1 - \sup_{y > r} \mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}\left(\frac{r-a}{b-a} + 1 - 1\right) = \frac{r-a}{2(b-a)}.$$

If $r \le a$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y \le r} \mu(y) + 1 - \sup_{y > r} \mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}(0 + 1 - 1) = 0.$$

That is,

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b, \\ \dfrac{c-2b+r}{2(c-b)}, & \text{if } b \le r \le c, \\ 1, & \text{if } c \le r. \end{cases}$$

On the similar lines, we can obtain

$$Cr\{\xi \ge r\} = \begin{cases} 1, & \text{if } r \le a, \\ \dfrac{2b-a-r}{2(b-a)}, & \text{if } a \le r \le b, \\ \dfrac{c-r}{2(c-b)}, & \text{if } b \le r \le c, \\ 0, & \text{if } c \le r. \end{cases}$$

A graphical representation of the credibility of events $\xi \le r$ (see the left part of Fig. 5.2) and $\xi \ge r$ (see the right part of Fig. 5.2) is presented in Fig. 5.2.

Definition 5.6 (Trapezoidal fuzzy variable). A fuzzy variable is said to be a *trapezoidal fuzzy variable* if it is fully determined by the quadruplet (a, b, c, d) of crisp numbers with $a < b < c < d$ and its membership function (see Fig. 5.3) is given by

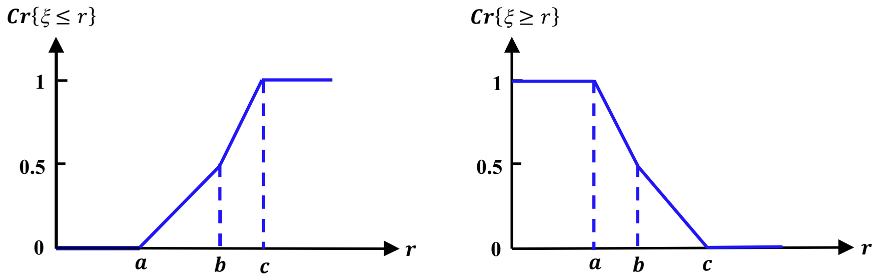

Fig. 5.2 Credibility of triangular fuzzy variable

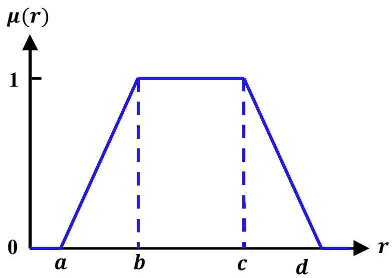

Fig. 5.3 Membership function of trapezoidal fuzzy variable

$$\mu(r) = \begin{cases} \dfrac{r-a}{b-a}, & \text{if } a \le r \le b\,, \\ 1, & \text{if } b \le r \le c\,, \\ \dfrac{d-r}{d-c}, & \text{if } c \le r \le d\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

To find the credibility $Cr\{\xi \le r\}$, we use the credibility inversion theorem. Accordingly, if $d \le r$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y \le r} \mu(y) + 1 - \sup_{y > r} \mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}(1 + 1 - 0) = 1\,.$$

If $c \le r \le d$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y\le r}\mu(y) + 1 - \sup_{y>r}\mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}\left(1 + 1 - \frac{d-r}{d-c}\right) = \frac{d-2c+r}{2(d-c)}.$$

If $b \le r \le c$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y\le r}\mu(y) + 1 - \sup_{y>r}\mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}(1 + 1 - 1) = \frac{1}{2}.$$

If $a \le r \le b$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y\le r}\mu(y) + 1 - \sup_{y>r}\mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}\left(\frac{r-a}{b-a} + 1 - 1\right) = \frac{r-a}{2(b-a)}.$$

If $r \le a$, we have

$$Cr\{\xi \le r\} = \frac{1}{2}\left(\sup_{y\le r}\mu(y) + 1 - \sup_{y>r}\mu(y)\right), \quad \forall r \in \mathbf{R}$$
$$\Rightarrow Cr\{\xi \le r\} = \frac{1}{2}(0 + 1 - 1) = 0.$$

That is,

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b, \\ \dfrac{1}{2}, & \text{if } b \le r \le c, \\ \dfrac{d-2c+r}{2(d-c)}, & \text{if } c \le r \le d, \\ 1, & \text{if } d \le r. \end{cases}$$

Similarly, we can obtain

$$Cr\{\xi \geq r\} = \begin{cases} 1, & \text{if } r \leq a\,, \\ \dfrac{2b-r-a}{2(b-a)}, & \text{if } a \leq r \leq b\,, \\ \dfrac{1}{2}, & \text{if } b \leq r \leq c\,, \\ \dfrac{d-r}{2(d-c)}, & \text{if } c \leq r \leq d\,, \\ 0, & \text{if } d \leq r\,. \end{cases}$$

A graphical representation of the credibility of events $\xi \leq r$ (see the left part of Fig. 5.4) and $\xi \geq r$ (see the right part of Fig. 5.4) is presented in Fig. 5.4.

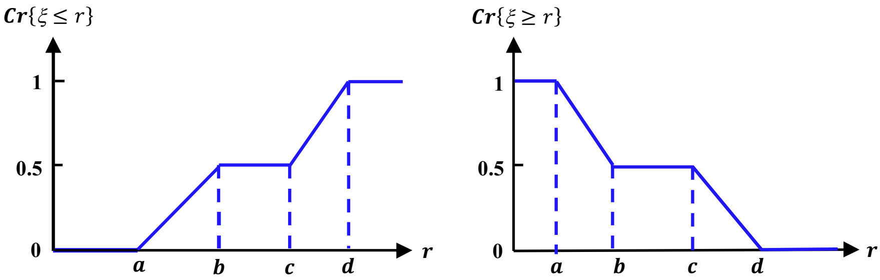

Fig. 5.4 Credibility of trapezoidal fuzzy variable

Expected Value of Fuzzy Variable

In literature, there are many ways to define an expected value operator for fuzzy variables, for example, Campos and González [15], Dubois and Prade [27], Heilpern [48] and Yager [121]. The most general definition of expected value operator of fuzzy variable was given by Liu and Liu [85]. This definition has an advantage in terms of its applicability, i.e., it is applicable not only to continuous fuzzy variables but also to discrete ones.

Definition 5.7 (Expected value). Let ξ be a fuzzy variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} Cr\{\xi \geq r\}dr - \int_{-\infty}^{0} Cr\{\xi \leq r\}dr \tag{5.4}$$

provided that at least one of the two integrals is finite.

The next two theorems provides the expected value of a triangular fuzzy variable and a trapezoidal fuzzy variable.

Theorem 5.4. *Let* $\xi = (a, b, c)$ *with* $a < b < c$ *be a triangular fuzzy variable. The* $E[\xi]$ *is given by*

$$E[\xi] = \frac{a + 2b + c}{4}.$$

Proof. From the credibility inversion theorem, we have

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a, \\ \dfrac{r - a}{2(b - a)}, & \text{if } a \le r \le b, \\ \dfrac{c - 2b + r}{2(c - b)}, & \text{if } b \le r \le c, \\ 1, & \text{if } c \le r. \end{cases}$$

and

$$Cr\{\xi \ge r\} = \begin{cases} 1, & \text{if } r \le a, \\ \dfrac{2b - a - r}{2(b - a)}, & \text{if } a \le r \le b, \\ \dfrac{c - r}{2(c - b)}, & \text{if } b \le r \le c, \\ 0, & \text{if } c \le r. \end{cases}$$

Now, we consider four possible cases to prove the theorem.
Case 1: If $a \ge 0$, then $Cr\{\xi \le r\} \equiv 0$ when $r < 0$ and

$$Cr\{\xi \ge r\} = \begin{cases} 1, & \text{if } r \le a, \\ \dfrac{2b - a - r}{2(b - a)}, & \text{if } a \le r \le b, \\ \dfrac{c - r}{2(c - b)}, & \text{if } b \le r \le c, \\ 0, & \text{if } c \le r. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] &= \left(\int_0^a 1dr + \int_a^b \frac{2b - a - r}{2(b - a)}dr + \int_b^c \frac{c - r}{2(c - b)}dr + \int_c^{+\infty} 0dr\right) \\ &\quad - \int_{-\infty}^0 0dr \\ &= \frac{a + 2b + c}{4}. \end{aligned}$$

Case 2: If $a < 0 \le b < c$, then

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{-2a}, & \text{if } a \le r \le 0\,, \end{cases}$$

$$Cr\{\xi \ge r\} = \begin{cases} \dfrac{2b-r}{2b}, & \text{if } 0 \le r \le b\,, \\ \dfrac{c-r}{2(c-b)}, & \text{if } b \le r \le c\,, \\ 0, & \text{if } c \le r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] &= \left(\int_0^b \frac{2b-r}{2b}dr + \int_b^c \frac{c-r}{2(c-b)}dr + \int_c^{+\infty} 0dr\right) - \left(\int_{-\infty}^a 0dr + \int_a^0 \frac{r-a}{-2a}dr\right) \\ &= \frac{a+2b+c}{4}\,. \end{aligned}$$

Case 3: If $a < b \le 0 < c$, then

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{r-2b}{-2b}, & \text{if } b \le r \le 0\,, \end{cases}$$

$$Cr\{\xi \ge r\} = \begin{cases} \dfrac{c-r}{2c}, & \text{if } 0 \le r \le c\,, \\ 0, & \text{if } c \le r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] &= \left(\int_0^c \frac{c-r}{2c}dr + \int_c^{+\infty} 0dr\right) - \left(\int_{-\infty}^a 0dr + \int_a^b \frac{r-a}{2(b-a)}dr + \int_b^0 \frac{r-2b}{-2b}dr\right) \\ &= \frac{a+2b+c}{4}\,. \end{aligned}$$

Case 4: If $a < b < c \le 0$, then $Cr\{\xi \ge r\} \equiv 0$ when $r > 0$ and

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{r+c-2b}{2(c-b)}, & \text{if } b \le r \le c\,, \\ 1, & \text{if } c \le r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$E[\xi] = \int_0^{+\infty} 0dr - \left(\int_{-\infty}^{a} 0dr + \int_a^b \frac{r-a}{2(b-a)}dr + \int_b^c \frac{r+c-2b}{2(c-b)}dr + \int_c^0 1dr \right)$$
$$= \frac{a+2b+c}{4}\,.$$

□

Theorem 5.5. *Let $\xi = (a, b, c, d)$ with $a < b < c < d$ be a trapezoidal fuzzy variable. The $E[\xi]$ is given by*

$$E[\xi] = \frac{a+b+c+d}{4}\,. \tag{5.5}$$

Proof. From the credibility inversion theorem, we have

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{1}{2}, & \text{if } b \le r \le c\,, \\ \dfrac{d-2c+r}{2(d-c)}, & \text{if } c \le r \le d\,, \\ 1, & \text{if } d \le r\,. \end{cases}$$

and

$$Cr\{\xi \ge r\} = \begin{cases} 1, & \text{if } r \le a\,, \\ \dfrac{2b-r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{1}{2}, & \text{if } b \le r \le c\,, \\ \dfrac{d-r}{2(d-c)}, & \text{if } c \le r \le d\,, \\ 0, & \text{if } d \le r\,. \end{cases}$$

Now, we consider five possible cases to prove the theorem.
Case 1: If $a \geq 0$, then $Cr\{\xi \leq r\} \equiv 0$ when $r < 0$ and

$$Cr\{\xi \geq r\} = \begin{cases} 1, & \text{if } r \leq a\,, \\ \dfrac{2b-a-r}{2(b-a)}, & \text{if } a \leq r \leq b\,, \\ \dfrac{1}{2}, & \text{if } b \leq r \leq c\,, \\ \dfrac{d-r}{2(d-c)}, & \text{if } c \leq r \leq d\,, \\ 0, & \text{if } d \leq r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] &= \left(\int_0^a 1dr + \int_a^b \frac{2b-a-r}{2(b-a)}dr + \int_b^c \frac{1}{2}dr + \int_c^d \frac{d-r}{2(d-c)}dr + \int_d^{+\infty} 0dr \right) \\ &\quad - \int_{-\infty}^0 0dr \\ &= \frac{a+b+c+d}{4}\,. \end{aligned}$$

Case 2: If $a < 0 \leq b < c < d$, then

$$Cr\{\xi \leq r\} = \begin{cases} 0, & \text{if } r \leq a\,, \\ \dfrac{r-a}{-2a}, & \text{if } a \leq r \leq 0\,, \end{cases}$$

$$Cr\{\xi \geq r\} = \begin{cases} \dfrac{2b-r}{2b}, & \text{if } 0 \leq r \leq b\,, \\ \dfrac{1}{2}, & \text{if } b \leq r \leq c\,, \\ \dfrac{d-r}{2(d-c)}, & \text{if } c \leq r \leq d\,, \\ 0, & \text{if } d \leq r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] &= \left(\int_0^b \frac{2b-r}{2b}dr + \int_b^c \frac{1}{2}dr + \int_c^d \frac{d-r}{2(d-c)}dr + \int_d^{+\infty} 0dr \right) \\ &\quad - \left(\int_{-\infty}^a 0dr + \int_a^0 \frac{r-a}{-2a}dr \right) \\ &= \frac{a+b+c+d}{4}\,. \end{aligned}$$

Case 3: If $a < b \le 0 < c < d$, then

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{1}{2}, & \text{if } b \le r \le 0\,, \end{cases}$$

$$Cr\{\xi \ge r\} = \begin{cases} \dfrac{1}{2}, & \text{if } 0 \le r \le c\,, \\ \dfrac{d-r}{2(d-c)}, & \text{if } c \le r \le d\,, \\ 0, & \text{if } d \le r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$\begin{aligned} E[\xi] = &\left(\int_0^c \frac{1}{2}dr + \int_c^d \frac{d-r}{2(d-c)}dr + \int_d^{+\infty} 0dr\right) \\ &-\left(\int_{-\infty}^a 0dr + \int_a^b \frac{r-a}{2(b-a)}dr + \int_b^0 \frac{1}{2}dr\right) \\ = &\frac{a+b+c+d}{4}\,. \end{aligned}$$

Case 4: If $a < b < c \le 0 < d$, then

$$Cr\{\xi \le r\} = \begin{cases} 0, & \text{if } r \le a\,, \\ \dfrac{r-a}{2(b-a)}, & \text{if } a \le r \le b\,, \\ \dfrac{1}{2}, & \text{if } b \le r \le c\,, \\ \dfrac{r-2c}{-2c}, & \text{if } c \le r \le 0\,, \end{cases}$$

$$Cr\{\xi \ge r\} = \begin{cases} \dfrac{d-r}{2d}, & \text{if } 0 \le r \le d\,, \\ 0, & \text{if } d \le r\,. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$E[\xi] = \left(\int_0^d \frac{d-r}{2d}dr + \int_d^{+\infty} 0dr\right) - \left(\int_{-\infty}^a 0dr + \int_a^b \frac{r-a}{2(b-a)}dr + \int_b^c \frac{1}{2}dr + \int_c^0 \frac{r-2c}{-2c}dr\right)$$
$$= \frac{a+b+c+d}{4}.$$

Case 5: If $a < b < c < d \leq 0$, then $Cr\{\xi \geq r\} \equiv 0$ when $r > 0$ and

$$Cr\{\xi \leq r\} = \begin{cases} 0, & \text{if } r \leq a, \\ \frac{r-a}{2(b-a)}, & \text{if } a \leq r \leq b, \\ \frac{1}{2}, & \text{if } b \leq r \leq c, \\ \frac{r-2c+d}{2(d-c)}, & \text{if } c \leq r \leq d, \\ 1, & \text{if } d \leq r. \end{cases}$$

Then the expected value of ξ using (5.4) is obtained as

$$E[\xi] = \int_0^{+\infty} 0dr - \left(\int_{-\infty}^a 0dr + \int_a^b \frac{r-a}{2(b-a)}dr + \int_b^c \frac{1}{2}dr + \int_c^d \frac{r-2c+d}{2(d-c)}dr + \int_d^0 1dr\right)$$
$$= \frac{a+b+c+d}{4}.$$

□

Variance of a Fuzzy Variable

The variance of a fuzzy variable provides a measure of the spread of the distribution around its expected value. A small value of variance indicates that the fuzzy variable is tightly concentrated around its expected value and a large value of variance indicates that the fuzzy variable has a wide spread around its expected value.

Definition 5.8 (Variance). Let ξ be a fuzzy variable with finite expected value e. Then the variance of ξ is defined by

$$V[\xi] = E[(\xi - e)^2].$$

Note that according to the above definition, the variance is simply the expected value of $(\xi - e)^2$. Since $(\xi - e)^2$ is a nonnegative uncertain variable, we also have

$$V[\xi] = \int_0^{+\infty} Cr\{(\xi - e)^2 \geq r\} dr .$$

The following theorems proved in [85] provides the variance of special fuzzy variables, namely, triangular and trapezoidal fuzzy variables.

Theorem 5.6. *Let $\xi = (a, b, c)$ with $a < b < c$ be a triangular fuzzy variable. The $V[\xi]$ is given by*

$$V[\xi] = \frac{33\alpha^3 + 21\alpha^2\beta + 11\alpha\beta^2 - \beta^3}{384\alpha},$$

where $\alpha = \max\{b-a, c-b\}$ and $\beta = \min\{b-a, c-b\}$. Further, when $\xi = (a, b, c)$ is a symmetric triangular fuzzy variable, i.e., $c - b = b - a$, then

$$V[\xi] = \frac{(c-a)^2}{24} .$$

Theorem 5.7. *Let $\xi = (a, b, c, d)$ with $a < b < c < d$ be a symmetric trapezoidal fuzzy variable, i.e., $d - c = b - a$ Then $V[\xi]$ is given by*

$$V[\xi] = \frac{((d-a)^2 + (d-a)(c-b) + (c-b)^2)}{24} . \tag{5.6}$$

The proof of these theorems can be worked out on similar lines as presented before for calculation of expected value.

5.2 Portfolio Selection Based on Credibility Theory

Considering that there are many non-probabilistic factors that may affect the asset returns, we assume that the investor allocate his/her wealth among n assets that offer fuzzy returns. We first introduce the following notation.

5.2.1 Notation

r_i: the fuzzy rate of return of the i-th asset,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio,} \\ 0, & \text{otherwise,} \end{cases}$$

L_i: the fuzzy turnover rate of the i-th asset,

L: the minimum desired level of the expected liquidity of the portfolio,

β: the minimum acceptable confidence level for the satisfaction of chance constraint corresponding to portfolio liquidity,

u_i: the maximal fraction of the capital allocated to the i-th asset,

l_i: the minimal fraction of the capital allocated to the i-th asset,

h: the number of assets held in the portfolio.

We consider the following objective functions and constraints in the bi-objective portfolio selection problem.

5.2.2 Objective Functions

Return

The expected return of the portfolio is expressed as

$$f_1(x) = E[r_1x_1 + r_2x_2 + \ldots + r_nx_n].$$

Risk

The portfolio risk is measured using variance in which the deviations of the portfolio return from the expected portfolio return contribute towards risk. That is, if the deviations are large, then the expected portfolio return is difficult to obtain. The portfolio risk is expressed as

$$f_2(x) = V[r_1x_1 + r_2x_2 + \ldots + r_nx_n].$$

5.2.3 Constraints

Liquidity

The chance constraint for the credibility of the fuzzy event that the portfolio liquidity not less than L is more than or equal to a confidence level β ($0.5 < \beta \leq 1$) is expressed as

$$Cr\{L_1x_1 + L_2x_2 + \ldots + L_nx_n \geq L\} \geq \beta.$$

Note that any confidence level $\beta \leq 0.5$ is considered low and thus may be regarded as meaningless to the real-world problems.

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1.$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \leq u_i y_i, \quad i = 1, 2, \ldots, n.$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,.$$

Number of assets held in the portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h\,.$$

No short selling of assets is expressed as

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

5.2.4 The Decision Problem

The bi-objective credibility-based fuzzy optimization model for portfolio selection problem is formulated as

$$\begin{aligned}
\mathbf{P(5.1)}\quad & \max\ f_1(x) = E[r_1x_1 + r_2x_2 + \ldots + r_nx_n] \\
& \min\ f_2(x) = V[r_1x_1 + r_2x_2 + \ldots + r_nx_n] \\
& \text{subject to} \\
& Cr\{L_1x_1 + L_2x_2 + \ldots + L_nx_n \geq L\} \geq \beta\,, && (5.7) \\
& \sum_{i=1}^{n} x_i = 1\,, && (5.8) \\
& \sum_{i=1}^{n} y_i = h\,, && (5.9) \\
& x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,, && (5.10) \\
& x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,, && (5.11) \\
& y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,, && (5.12) \\
& x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,. && (5.13)
\end{aligned}$$

5.3 Solution Methodology

The model P(5.1) is difficult to solve analytically in the presence of fuzzy variables; thus, we use a two-phase approach. In the first phase, the model P(5.1) is converted into an equivalent crisp bi-objective model. Then, in the second phase, an fuzzy interactive approach is used for finding a preferred compromise solution through an interaction between the investor and model analyzer.

5.3.1 First Phase: Crisp Equivalent Bi-objective Model

Here, we present the appropriate strategies for obtaining the crisp equivalent of the model P(5.1). We assume that asset returns and turnover rates are the independent symmetrical trapezoidal fuzzy variables denoted by $r_i = (r_{a_i}, r_{b_i}, r_{c_i}, r_{d_i})$ and $L_i = (L_{a_i}, L_{b_i}, L_{c_i}, L_{d_i})$ with $r_{b_i} - r_{a_i} = r_{d_i} - r_{c_i}$ and $L_{b_i} - L_{a_i} = L_{d_i} - L_{c_i},\ i = 1, 2, \ldots, n$, respectively. It is worthy to mention that, we may also consider general fuzzy variables for asset returns and turnover rates (other than linear, triangular or trapezoidal) but then it is computationally difficulty to obtain the crisp equivalent of the fuzzy optimization model for portfolio selection. Note that trapezoidal fuzzy variables are most widely-used in real-world problems and are easy-processed fuzzy variables. Further, triangular fuzzy variables are the special type of trapezoidal fuzzy variables.

- **Treating the objective functions**

Using equations (5.5) and (5.6), the return and risk objective functions of the model P(5.1) are replaced by the following crisp objectives, respectively.

$$\max f_1(x) = \sum_{i=1}^{n} \frac{r_{a_i} + r_{b_i} + r_{c_i} + r_{d_i}}{4} x_i ,$$

$$\min f_2(x) = \sum_{i,j=1}^{n} \left(\frac{(r_{d_i} - r_{a_i})(r_{d_j} - r_{a_j}) + (r_{d_i} - r_{a_i})(r_{c_j} - r_{b_j}) + (r_{c_i} - r_{b_i})(r_{c_j} - r_{b_j})}{24} \right) x_i x_j .$$

- **Treating the chance constraint**

To deal with the chance constraint in respect of portfolio liquidity, we use a strategy based on the following result established in [131].

Theorem 5.8. *Let $\xi = (a, b, c, d)$ with $a < b < c < d$ be a fuzzy number. For a given confidence level λ $(0.5 < \lambda \leq 1)$, we have*

$$Cr\{\xi \geq r\} \geq \lambda \Leftrightarrow r \leq (2\lambda - 1)a + 2(1 - \lambda)b . \tag{5.14}$$

Proof. According to the credibility distribution $Cr\{\xi \geq r\}$, we can deduce that if $0.5 < \lambda \leq 1$ then the following two cases may be considered:

Case 1: $Cr\{\xi \geq r\} \geq \lambda \Rightarrow r \leq a (Cr\{\xi \geq r\} \equiv 1)$ or $\dfrac{2b - a - r}{2(b - a)} \geq \lambda$. If $r \leq a$, then $r \leq a \leq (2\lambda - 1)a + 2(1 - \lambda)b$ under the condition that $0.5 < \lambda \leq 1$; If $\dfrac{2b - a - r}{2(b - a)} \geq \lambda$, then $r \leq (2\lambda - 1)a + 2(1 - \lambda)b$.

Case 2: If $r \leq (2\lambda - 1)a + 2(1 - \lambda)b$ then $\dfrac{2b - a - r}{2(b - a)} \geq \lambda \Rightarrow Cr\{\xi \geq r\} \geq \lambda$. Hence, $Cr\{\xi \geq r\} \geq \lambda \Leftrightarrow r \leq (2\lambda - 1)a + 2(1 - \lambda)b$. □

Using (5.14), the chance constraint (5.7) of the model P(5.1) is replaced by the following crisp constraint:

$$\sum_{i=1}^{n} ((2\beta - 1)L_{a_i} + (2 - 2\beta)L_{b_i})x_i \geq L\,. \tag{5.15}$$

Consequently, the model P(5.1) is converted into the following equivalent crisp model:

P(5.2) $\max f_1(x) = \sum_{i=1}^{n} \dfrac{r_{a_i} + r_{b_i} + r_{c_i} + r_{d_i}}{4} x_i$

$\min f_2(x) = \sum_{i,j=1}^{n} \left(\dfrac{(r_{d_i} - r_{a_i})(r_{d_j} - r_{a_j}) + (r_{d_i} - r_{a_i})(r_{c_j} - r_{b_j}) + (r_{c_i} - r_{b_i})(r_{c_j} - r_{b_j})}{24} \right) x_i x_j$

subject to

Constraints (5.8) – (5.13) and (5.15) .

5.3.2 Second Phase: Fuzzy Interactive Approach

To handle the bi-objective mixed integer quadratic programming problem P(5.2), we present an fuzzy interactive approach based on the TH aggregation function [115]. It is noteworthy to mention that unlike the existing methods such as Guu and Wu [46], Lai and Hwang [75], Sakawa, Yano and Yumine [103] that may lead to weakly efficient solutions, the TH method guarantees to find efficient solutions.

The solution methodology of the fuzzy interactive approach for model P(5.2) consists of the following steps:

Step 1: Solve the problem P(5.2) as a single-objective problem in respect of return and risk objective functions. Mathematically,

For return objective function:

$\max f_1(x)$ subject to constraints (5.8)-(5.13) and (5.15).

For risk objective function:

$\min f_2(x)$ subject to constraints (5.8)-(5.13) and (5.15).

Let x^1 and x^2 denote the optimal solutions obtained by solving the single-objective problems in respect of return and risk objective functions, respectively. If both the solutions, i.e., $x^1 = x^2 = (x_1, x_2, \ldots, x_n)$ are same, we obtain an efficient (preferred compromise) solution and stop; otherwise, go to Step 2.

Step 2: Evaluate both the objective functions at the obtained solutions. Determine the worst lower bound (f_1^L) and best upper bound (f_1^R) for return objective; and, the best lower bound (f_2^L) and worst upper bound (f_2^R) for risk objective as follows:

$$f_1^R = f_1(x^1),$$
$$f_1^L = f_1(x^2),$$
$$f_2^L = f_2(x^2),$$
$$f_2^R = f_2(x^1).$$

Step 3: Define the linear membership functions for return and risk objective functions as follows:

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq f_1^R, \\ \frac{f_1(x)-f_1^L}{f_1^R-f_1^L}, & \text{if } f_1^L < f_1(x) < f_1^R, \\ 0, & \text{if } f_1(x) \leq f_1^L, \end{cases}$$

where $\mu_{f_1}(x)$ denotes the satisfaction degree of return objective function for the given solution x.

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \leq f_2^L, \\ \frac{f_2^R-f_2(x)}{f_2^R-f_2^L}, & \text{if } f_2^L < f_2(x) < f_2^R, \\ 0, & \text{if } f_2(x) \geq f_2^R, \end{cases}$$

where $\mu_{f_2}(x)$ denotes the satisfaction degree of risk objective function for the given solution x. A graphical representation of the above membership functions is presented in Figs. 5.5 and 5.6, respectively.

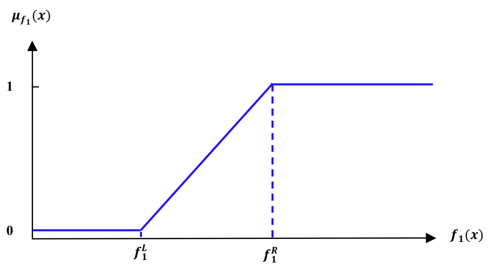

Fig. 5.5 The membership function of the return objective

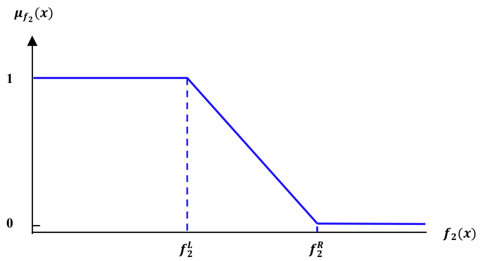

Fig. 5.6 The membership function of the risk objective

Step 4: Convert the crisp equivalent problem P(5.2) into a single-objective problem P(5.3) as follows:

$$\begin{aligned}
\textbf{P(5.3)}\quad & \max\ w\alpha + (1-w)(\gamma_1\mu_{f_1}(x) + \gamma_2\mu_{f_2}(x)) \\
& \text{subject to} \\
& \alpha \le \mu_{f_1}(x), \\
& \alpha \le \mu_{f_2}(x), \\
& 0 \le w \le 1, \\
& 0 \le \alpha \le 1, \\
& \text{and Constraints (5.8)} - \text{(5.13) and (5.15)},
\end{aligned}$$

where $\alpha = \min\{\mu_{f_1}(x), \mu_{f_2}(x)\}$ denotes the minimum satisfaction degree of objectives. This formulation has a new achievement function defined as a convex combination of the lower bound on the satisfaction levels of objectives, i.e., α and the weighted sum of the achievement degrees, i.e., $\mu_{f_1}(x)$ and $\mu_{f_2}(x)$ that ensure yielding an adjustably balanced compromise solution. Moreover, γ_1, γ_2 indicate the relative importance of the return and risk objective functions, respectively, and w indicate the relative importance of the coefficient of compensation. The parameters γ_1, γ_2 are determined using investor preferences in respect of the two objective functions such that $\gamma_1 + \gamma_2 = 1, \gamma_1, \gamma_2 > 0$. Also, w controls the minimum satisfaction level of objectives as well as the compromise degree between the objectives implicitly. Thus, the problem P(5.3) is capable of yielding both unbalanced and balanced compromise solutions based on the investor preferences through adjustment of the value of parameter w. To be more explanatory, a higher value for w indicates that more attention is paid to obtain a higher lower bound (α) for the satisfaction degree of the two objectives and accordingly more balanced compromise solutions are obtained. On the contrary, the lower value for w indicates that more attention is paid to obtain solutions with larger satisfaction degree for the objective that has higher relative importance, i.e., yielding unbalanced compromise solutions.

Step 5: Determine importance of the objectives (γ_1, γ_2) and the value of compensation coefficient (w) based on investor preferences and solve the resulting single-objective problem P(5.3). If the investor is satisfied with the obtained efficient solution, then stop and select the current solution as the final decision; otherwise, alter the required parameters such as $\beta, \gamma_1, \gamma_2$ and w according to the revised and updated preferences of the investor. Reformulate model P(5.2)/P(5.3) as the case may be and go to either Step 1 or Step 5.

Fig. 5.7 depicts the flowchart of the fuzzy interactive approach.

5.4 Numerical Illustration

We now discuss a real-world empirical study done for an imaginary investor using the data set extracted from NSE, Mumbai, India corresponding to ten different assets. Numerical tests are carried out using different confidence levels, importance weights of objective functions and variations in the compensation coefficient.

Since, it is assumed that the return and turnover rates for each asset are symmetrical trapezoidal fuzzy variables, we need to estimate these parameters. For this purpose, a group of experts has been formed to specify the four prominent values used to determine the corresponding trapezoidal fuzzy numbers based on the available historical data and experts knowledge. As an illustration, consider the calculation of fuzzy return of the asset AHB. The historical data (daily return from April 1, 2005 to March 31, 2008) is used to calculate the frequency of historical returns. Based upon experts knowledge and historical data, we find that most of the returns fall into the intervals [0.10246, 0.16350], [0.16350, 0.19454], [0.19454, 0.22558] and [0.22558, 0.27308]. We take the mid-points of the intervals [0.10246, 0.16350] and [0.22558, 0.27308] as the left and the right-end points of the tolerance interval, respectively. Thus, the tolerance interval of the fuzzy return becomes [0.13298, 0.24933]. Further, by observing all the historical data and experts opinion, we use 0.08311 and 0.29920 as the minimum possible value and the maximum possible value, respectively, of the uncertain return. Hence, the fuzzy return of the asset AHB becomes [0.08311, 0.13298, 0.24933, 0.29920]. Similarly, we obtain fuzzy return and turnover rates of the remaining assets. Table 5.1 gives the fuzzy data regarding return and liquidity.

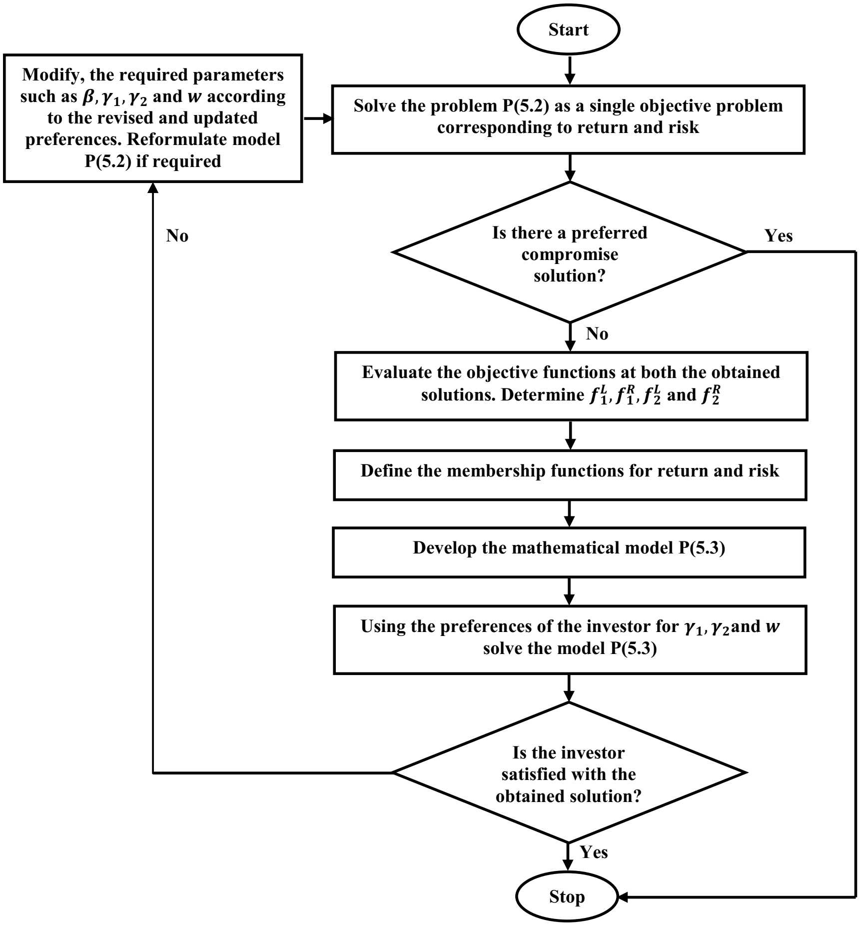

Fig. 5.7 Flow chart of the fuzzy interactive approach

Table 5.1 Input data of assets corresponding to fuzzy return and liquidity

Company	Return	Liquidity
Allahabad Bank (AHB)	(0.08311,0.13298,0.24933,0.29920)	(0.00151,0.00403,0.00755,0.01007)
Axis Bank Ltd. (ABL)	(0.05946,0.15855,0.29729,0.39638)	(0.00100,0.00267,0.00501,0.00668)
B E M L Ltd. (BML)	(0.08464,0.22570,0.42319,0.56425)	(0.00413,0.01102,0.02067,0.02756)
Bharti Airtel Ltd. (BAL)	(0.06961,0.18562,0.34804,0.46406)	(0.00130,0.00348,0.00652,0.00869)
Essar Oil Ltd. (EOL)	(0.07615,0.20306,0.38073,0.50765)	[0.00365,0.00973,0.01825,0.02433]
Gammon India Ltd. (GIL)	(0.08829,0.23543,0.44143,0.58857)	(0.00031,0.00083,0.00156,0.00208)
Jindal Saw Ltd. (JSL)	(0.04680,0.12481,0.23402,0.31203)	(0.00282,0.00751,0.01408,0.01877)
Sesa Goa Ltd. (SGL)	(0.09632,0.25685,0.48159,0.64212)	(0.00476,0.01268,0.02378,0.03170)
Tata Steel Ltd. (TSL)	(0.05440,0.14508,0.27202,0.36269)	(0.00333,0.00889,0.01666,0.02221)
United Spirits Ltd. (USL)	(0.08240,0.21974,0.41202,0.54936)	(0.00288,0.00767,0.01438,0.01918)

Optimal Asset Allocation

To find an optimal asset allocation, i.e., efficient (preferred compromise) solution, we first formulate the uncertain model P(5.1) using the input data from Table 5.1, $L = 0.0045$, $h = 5$, $l_i = 0.08$, and $u_i = 0.45,\ i = 1, 2, \ldots, 10$.

$$\begin{aligned}
\max\ f_1(x) = E\,[&(0.08311, 0.13298, 0.24933, 0.29920)x_1 + (0.05946, 0.15855, 0.29729, 0.39638)x_2\\
&+(0.08464, 0.22570, 0.42319, 0.56425)x_3 + (0.06961, 0.18562, 0.34804, 0.46406)x_4\\
&+(0.07615, 0.20306, 0.38073, 0.50765)x_5 + (0.08829, 0.23543, 0.44143, 0.58857)x_6\\
&+(0.04680, 0.12481, 0.23402, 0.31203)x_7 + (0.09632, 0.25685, 0.48159, 0.64212)x_8\\
&+(0.05440, 0.14508, 0.27202, 0.36269)x_9 + (0.08240, 0.21974, 0.41202, 0.54936)x_{10}]\\
\min\ f_2(x) = V\,[&(0.08311, 0.13298, 0.24933, 0.29920)x_1 + (0.05946, 0.15855, 0.29729, 0.39638)x_2\\
&+(0.08464, 0.22570, 0.42319, 0.56425)x_3 + (0.06961, 0.18562, 0.34804, 0.46406)x_4\\
&+(0.07615, 0.20306, 0.38073, 0.50765)x_5 + (0.08829, 0.23543, 0.44143, 0.58857)x_6\\
&+(0.04680, 0.12481, 0.23402, 0.31203)x_7 + (0.09632, 0.25685, 0.48159, 0.64212)x_8\\
&+(0.05440, 0.14508, 0.27202, 0.36269)x_9 + (0.08240, 0.21974, 0.41202, 0.54936)x_{10}]
\end{aligned}$$

subject to

$$\begin{aligned}
&Cr\{(0.00151, 0.00403, 0.00755, 0.01007)x_1 + (0.00100, 0.00267, 0.00501, 0.00668)x_2\\
&\quad+(0.00413, 0.01102, 0.02067, 0.02756)x_3 + (0.00130, 0.00348, 0.00652, 0.00869)x_4\\
&\quad+(0.00365, 0.00973, 0.01825, 0.02433)x_5 + (0.00031, 0.00083, 0.00156, 0.00208)x_6\\
&\quad+(0.00282, 0.00751, 0.01408, 0.01877)x_7 + (0.00476, 0.01268, 0.02378, 0.03170)x_8\\
&\quad+(0.00333, 0.00889, 0.01666, 0.02221)x_9 + (0.00288, 0.00767, 0.01438, 0.01918)x_{10} \geq 0.0045\}\\
&\geq \beta, \qquad (5.16)\\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1, \qquad (5.17)\\
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} = 5, \qquad (5.18)\\
&x_i - 0.08y_i \geq 0, \quad i = 1, 2, \ldots, 10, \qquad (5.19)\\
&x_i - 0.45y_i \leq 0, \quad i = 1, 2, \ldots, 10, \qquad (5.20)\\
&y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, 10, \qquad (5.21)\\
&x_i \geq 0, \quad i = 1, 2, \ldots, 10. \qquad (5.22)
\end{aligned}$$

Using the first phase of the solution methodology, the above model is converted into its crisp equivalent bi-objective model for a given minimum acceptable confidence level $\beta = 0.7$ as follows:

$$\begin{aligned}
\max\ f_1(x) = {}&0.19116x_1 + 0.22792x_2 + 0.32445x_3 + 0.26683x_4 + 0.29190x_5\\
&+0.33843x_6 + 0.17942x_7 + 0.36922x_8 + 0.20855x_9 + 0.31588x_{10}\\
\min\ f_2(x) = {}&0.00356x_1^2 + 0.00748x_2^2 + 0.01516x_3^2 + 0.01025x_4^2 + 0.01227x_5^2\\
&+0.01649x_6^2 + 0.00463x_7^2 + 0.01963x_8^2 + 0.00626x_9^2 + 0.01437x_{10}^2\\
&+0.004955x_1x_2 + 0.007054x_1x_3 + \ldots + 0.009485x_9x_{10}
\end{aligned}$$

subject to

$$\begin{aligned}
&0.00302x_1 + 0.00200x_2 + 0.00827x_3 + 0.00261x_4 + 0.00730x_5\\
&\quad+0.00062x_6 + 0.00563x_7 + 0.00951x_8 + 0.00666x_9 + 0.00575x_{10} \geq 0.0045,
\end{aligned}$$

and Constraints (5.17)-(5.22).

The above model is solved using the fuzzy interactive approach developed in Section 5.3.2. The models are coded and solved using LINGO 12.0.

Step 1: We first determine the worst lower (upper) bound and best upper (lower) bound for return and risk objective functions, respectively, by solving the following single-objective problems.

For Return Objective Function

$$\begin{aligned}\max\ f_1(x) = {}& 0.19116x_1 + 0.22792x_2 + 0.32445x_3 + 0.26683x_4 + 0.29190x_5 \\ & +0.33843x_6 + 0.17942x_7 + 0.36922x_8 + 0.20855x_9 + 0.31588x_{10}\end{aligned}$$

subject to

$$\begin{aligned}& 0.00302x_1 + 0.00200x_2 + 0.00827x_3 + 0.00261x_4 + 0.00730x_5 \\ & \quad +0.00062x_6 + 0.00563x_7 + 0.00951x_8 + 0.00666x_9 + 0.00575x_{10} \geq 0.0045,\end{aligned}$$

and Constraints (5.17)-(5.22).

The obtained solution is denoted as $x^1 = (x_1, x_2, \ldots, x_{10})$ and is provided in Table 5.2.

For Risk Objective Function

$$\begin{aligned}\min\ f_2(x) = {}& 0.00356x_1^2 + 0.00748x_2^2 + 0.01516x_3^2 + 0.01025x_4^2 + 0.01227x_5^2 \\ & +0.01649x_6^2 + 0.00463x_7^2 + 0.01963x_8^2 + 0.00626x_9^2 + 0.01437x_{10}^2 \\ & +0.004955x_1x_2 + 0.007054x_1x_3 + \ldots + 0.009485x_9x_{10}\end{aligned}$$

subject to

$$\begin{aligned}& 0.00302x_1 + 0.00200x_2 + 0.00827x_3 + 0.00261x_4 + 0.00730x_5 \\ & \quad +0.00062x_6 + 0.00563x_7 + 0.00951x_8 + 0.00666x_9 + 0.00575x_{10} \geq 0.0045,\end{aligned}$$

and Constraints (5.17)-(5.22).

The obtained solution is denoted as $x^2 = (x_1, x_2, \ldots, x_{10})$ and is provided in Table 5.2.

Table 5.2 The proportions of the assets in the obtained portfolio corresponding to single-objective problems

	Allocation				
	AHB	ABL	BML	BAL	EOL
x^1	0.0	0.0	0.08	0.0	0.08
x^2	0.32220	0.15335	0.0	0.0	0.09348
	GIL	JSL	SGL	TSL	USL
	0.31	0.0	0.45	0.0	0.08
	0.0	0.24774	0.0	0.18323	0.0

Step 2: Both the objective functions are evaluated at the obtained solutions, i.e., x^1 and x^2. The objective function values are provided in Table 5.3.

Table 5.3 Objective function values of return and risk

	x^1	x^2
Return ($f_1(x)$)	0.34564	0.20649
Risk ($f_2(x)$)	0.02381	0.00642

Now, we define the worst lower (upper) bound and best upper (lower) bound of both the objective functions as

$$0.20649 \leq f_1(x) \leq 0.34564\,,$$
$$0.00642 \leq f_2(x) \leq 0.02381\,.$$

Step 3: The linear membership function of the objective of expected return is

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq 0.34564\,, \\ \frac{f_1(x)-0.20649}{0.34564-0.20649}, & \text{if } 0.20649 < f_1(x) < 0.34564\,, \\ 0, & \text{if } f_1(x) \leq 0.20649\,, \end{cases}$$

and the linear membership function of the objective of risk is

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \leq 0.00642\,, \\ \frac{0.02381-f_2(x)}{0.02381-0.00642}, & \text{if } 0.00642 < f_2(x) < 0.02381\,, \\ 0, & \text{if } f_2(x) \geq 0.02381\,. \end{cases}$$

$$\begin{aligned}
&\max\ 0.2\alpha + 0.8(0.05\mu_{f_1}(x) + 0.95\mu_{f_2}(x)) \\
&\text{subject to} \\
&0.19116x_1 + 0.22792x_2 + 0.32445x_3 + 0.26683x_4 + 0.29190x_5 + 0.33843x_6 \\
&\quad +0.17942x_7 + 0.36922x_8 + 0.20855x_9 + 0.31588x_{10} - 0.13915\alpha \geq 0.20649, \\
&0.00356x_1^2 + 0.00748x_2^2 + 0.01516x_3^2 + 0.01025x_4^2 + 0.01227x_5^2 + 0.01649x_6^2 \\
&\quad +0.00463x_7^2 + 0.01963x_8^2 + 0.00626x_9^2 + 0.01437x_{10}^2 + 0.004955x_1x_2 \\
&\quad +0.007054x_1x_3 + \ldots + 0.009485x_9x_{10} + 0.01739\alpha \leq 0.02381, \\
&0.00302x_1 + 0.00200x_2 + 0.00827x_3 + 0.00261x_4 + 0.00730x_5 \\
&\quad +0.00062x_6 + 0.00563x_7 + 0.00951x_8 + 0.00666x_9 + 0.00575x_{10} \geq 0.0045, \\
&0 \leq \alpha \leq 1\,, \\
&\text{and Constraints (5.17)-(5.22)}\,.
\end{aligned}$$

Table 5.4 Summary results of portfolio selection

Minimum confidence level	Importance of first objective function	Satisfaction degrees		Objective function values	
β	$\gamma_1 = (1 - \gamma_2)$	$\mu_{f_1}(x)$	$\mu_{f_2}(x)$	Return ($f_1(x)$)	Risk ($f_2(x)$)
0.7	0.05	0.03361	0.99533	0.21117	0.00650

Table 5.5 The proportions of the assets in the obtained portfolio

	Allocation				
	AHB	ABL	BML	BAL	EOL
Portfolio	0.29933	0.16856	0.0	0.0	0.13044
	GIL	JSL	SGL	TSL	USL
	0.0	0.21669	0.0	0.18498	0.0

The computational result is summarized in Table 5.4. Table 5.5 presents proportions of the assets in the obtained portfolio.

Suppose the investor is not satisfied with the obtained portfolio. More portfolios may be generated by varying the values of minimum confidence level (β) and the importance weights of objective functions (γ_1, γ_2) according to investor preferences. The corresponding computational results are summarized in Table 5.6. Table 5.7 presents proportions of the assets in the obtained portfolios. Note that the compensation coefficient (w) is set to 0.2 in all numerical tests performed corresponding to the results summarized in Tables 5.6-5.7.

It can be seen from the results reported in Table 5.6 (see the fifth and sixth columns of Table 5.6) that the relationship between return and risk objective functions follows risk-return trade-off principle, i.e., an increase in return objective value leads to an increase in risk objective value and vice versa. It may be noted that the risk objective values listed in Table 5.6 are based on variance measure. Also, the computational results clearly shows that the risk-return trade-off is widen when the importance weight of the return objective function is either $\gamma_1 < 0.25$ or $\gamma_1 > 0.55$. On the other hand, more balanced solutions are obtained when $0.25 \leq \gamma_1 \leq 0.45$. In order to exemplify, from Table 5.6, for $\beta = 0.7, \gamma_1 = 0.05$, we have $\mu_{f_1}(x) = 0.03361$ and $\mu_{f_2}(x) = 0.99533$. However, for the same β, i.e., $\beta = 0.7, \gamma_1 = 0.45$, we have $\mu_{f_1}(x) = 0.69113$ and $\mu_{f_2}(x) = 0.69128$. This is because all the numerical

Table 5.6 Summary results of portfolio selection

Minimum confidence level	Importance of first objective function	Satisfaction degrees		Objective function values	
β	$\gamma_1 = (1 - \gamma_2)$	$\mu_{f_1}(x)$	$\mu_{f_2}(x)$	Return ($f_1(x)$)	Risk ($f_2(x)$)
0.7	0.05	0.03361	0.99533	0.21117	0.00650
	0.1	0.30143	0.89915	0.24844	0.00817
	0.15	0.53186	0.80372	0.28050	0.00983
	0.25	0.59826	0.76595	0.28974	0.01049
	0.35	0.68505	0.69694	0.30182	0.01169
	0.45	0.69113	0.69128	0.30267	0.01179
	0.55	0.86450	0.58798	0.32679	0.01340
	0.65	0.87186	0.58108	0.32782	0.01360
	0.75	0.88222	0.56771	0.32926	0.01374
	0.85	0.89771	0.53961	0.33142	0.01437
	0.95	0.92358	0.47120	0.33502	0.01552
0.8	0.05	0.05973	0.99011	0.22199	0.00659
	0.1	0.31193	0.89936	0.25516	0.00822
	0.15	0.46226	0.83868	0.27493	0.00923
	0.25	0.62168	0.74547	0.29589	0.01085
	0.35	0.68629	0.69410	0.30439	0.01184
	0.45	0.81340	0.60434	0.32609	0.01330
	0.55	0.85741	0.60025	0.32689	0.01357
	0.65	0.86548	0.59269	0.32795	0.01371
	0.75	0.87687	0.57802	0.32945	0.01380
	0.85	0.89384	0.54723	0.33168	0.01429
	0.95	0.92215	0.47228	0.33540	0.01560
0.9	0.05	0.12474	0.97972	0.25041	0.00677
	0.1	0.28243	0.92608	0.26738	0.00837
	0.15	0.29088	0.92246	0.26829	0.00847
	0.25	0.70397	0.69686	0.31274	0.01186
	0.35	0.70640	0.69608	0.31301	0.01196
	0.45	0.70937	0.69467	0.31333	0.01201
	0.55	0.83521	0.61488	0.32687	0.01347
	0.65	0.84489	0.60682	0.33791	0.01582
	0.75	0.85852	0.58824	0.33838	0.01590
	0.85	0.87891	0.55127	0.33957	0.01593
	0.95	0.91290	0.46134	0.34021	0.01598

Table 5.7 The proportions of the assets in the obtained portfolios

Minimum confidence level	Importance of first objective function	Allocation									
β	$\gamma_1 = (1 - \gamma_2)$	AHB	ABL	BML	BAL	EOL	GIL	JSL	SGL	TSL	USL
0.7	0.05	0.29933	0.16856	0.0	0.0	0.13044	0.0	0.21669	0.0	0.18498	0.0
	0.1	0.35043	0.0	0.13979	0.0	0.0	0.13405	0.25286	0.12287	0.0	0.0
	0.15	0.38265	0.0	0.15943	0.0	0.0	0.15328	0.0	0.14117	0.0	0.016347
	0.25	0.32043	0.0	0.17381	0.0	0.0	0.16929	0.0	0.15980	0.0	0.17668
	0.35	0.23907	0.0	0.19261	0.0	0.0	0.19023	0.0	0.18415	0.0	0.19394
	0.45	0.23337	0.0	0.19393	0.0	0.0	0.19170	0.0	0.18586	0.0	0.19514
	0.55	0.0	0.0	0.20104	0.0	0.213345	0.19602	0.0	0.18539	0.0	0.20420
	0.65	0.0	0.0	0.20116	0.0	0.20021	0.20038	0.0	0.19690	0.0	0.20135
	0.75	0.0	0.0	0.20133	0.0	0.18184	0.20647	0.0	0.21301	0.0	0.19735
	0.85	0.0	0.0	0.20158	0.0	0.15427	0.21562	0.0	0.23717	0.0	0.19136
	0.95	0.0	0.0	0.20201	0.0	0.10833	0.23086	0.0	0.27745	0.0	0.18135
0.8	0.05	0.31621	0.0	0.12260	0.0	0.13690	0.0	0.22919	0.0	0.19510	0.0
	0.1	0.35897	0.0	0.14384	0.0	0.0	0.0	0.0	0.12653	0.22295	0.14771
	0.15	0.36601	0.0	0.15800	0.0	0.17348	0.0	0.0	0.14071	0.0	0.16180
	0.25	0.28010	0.0	0.19744	0.0	0.0	0.15171	0.0	0.18439	0.0	0.18636
	0.35	0.22202	0.0	0.19943	0.0	0.0	0.18899	0.0	0.19177	0.0	0.19779
	0.45	0.0	0.0	0.20095	0.0	0.22230	0.19305	0.0	0.17754	0.0	0.20616
	0.55	0.0	0.0	0.20105	0.0	0.21210	0.19643	0.0	0.18647	0.0	0.20395
	0.65	0.0	0.0	0.20117	0.0	0.19851	0.20094	0.0	0.19839	0.0	0.20099
	0.75	0.0	0.0	0.20135	0.0	0.17948	0.20726	0.0	0.21508	0.0	0.19683
	0.85	0.0	0.0	0.20161	0.0	0.15093	0.21673	0.0	0.24010	0.0	0.19063
	0.95	0.0	0.0	0.20205	0.0	0.10335	0.23251	0.0	0.28181	0.0	0.18028
0.9	0.05	0.0	0.0	0.14604	0.0	0.16611	0.0	0.30904	0.12510	0.25371	0.0
	0.1	0.21531	0.0	0.17753	0.0	0.18623	0.0	0.0	0.16482	0.25611	0.0
	0.15	0.21931	0.0	0.18012	0.0	0.18813	0.0	0.0	0.16806	0.24438	0.0
	0.25	0.0	0.0	0.20994	0.18573	0.23376	0.0	0.0	0.18347	0.0	0.18710
	0.35	0.0	0.0	0.20913	0.18619	0.22859	0.0	0.0	0.18653	0.0	0.18956
	0.45	0.0	0.0	0.20811	0.18677	0.22211	0.0	0.0	0.19035	0.0	0.19266
	0.55	0.0	0.0	0.22264	0.0	0.22653	0.13969	0.0	0.21304	0.0	0.19810
	0.65	0.0	0.0	0.22272	0.0	0.21315	0.14422	0.0	0.22469	0.0	0.19522
	0.75	0.0	0.0	0.22284	0.0	0.19443	0.15056	0.0	0.24101	0.0	0.19116
	0.85	0.0	0.0	0.22301	0.0	0.16635	0.16008	0.0	0.26548	0.0	0.18508
	0.95	0.0	0.0	0.22331	0.0	0.11955	0.17594	0.0	0.30626	0.0	0.17494

tests have been performed for a low value of compensation coefficient, i.e., $w = 0.2$ where more attention is paid to obtain solutions that have larger satisfaction degree for the objective with higher relative importance, i.e., yielding unbalanced compromise solutions.

Next, suppose the investor desires that more attention be paid to obtain higher satisfaction degrees for both the objectives and accordingly more balanced compromise solutions. To do so, we set a higher value for compensation coefficient, $w = 0.8$, in the single-objective model formulated above. Numerical tests are carried out using different confidence levels and importance weights of objective functions. The computational results are summarized in Table 5.8 (see the first and second columns of Table 5.8). Table 5.9 presents proportions of the assets in the obtained portfolios.

Table 5.8 Summary results of portfolio selection

Minimum confidence level	Importance of first objective function	Satisfaction degrees		Objective function values	
β	$\gamma_1 = (1 - \gamma_2)$	$\mu_{f_1}(x)$	$\mu_{f_2}(x)$	Return ($f_1(x)$)	Risk ($f_2(x)$)
0.7	0.05-0.95	0.69113	0.69128	0.30267	0.01179
0.8	0.05-0.95	0.69027	0.69039	0.30491	0.01181
0.9	0.05-0.95	0.69986	0.69738	0.31230	0.01183

It can be seen from the results reported in Table 5.8 that for a given confidence level (β), only a single balanced compromise solution is obtained corresponding to different importance weights of the return objective function (γ_1). To be more explanatory, a higher value for compensation coefficient ($w = 0.8$) corresponds to more attention being paid in obtaining a higher lower bound (α) for the satisfaction degree of the objectives with no attention paid to the relative importance of the individual objective functions. Hence, the relative importance weights of both the objective functions do not significantly contribute in the achievement of the solutions.

The computational results presented in Tables 5.6 and 5.8 clearly shows that the fuzzy interactive approach has many advantages that makes it more realistic and flexible:

Table 5.9 The proportions of the assets in the obtained portfolios

Minimum confidence level	Importance of first objective function	Allocation									
β	$\gamma_1 = (1 - \gamma_2)$	AHB	ABL	BML	BAL	EOL	GIL	JSL	SGL	TSL	USL
0.7	0.05-0.95	0.23337	0.0	0.19393	0.0	0.0	0.19170	0.0	0.18586	0.0	0.19514
0.8	0.05-0.95	0.21844	0.0	0.19955	0.0	0.0	0.19129	0.0	0.19223	0.0	0.19849
0.9	0.05-0.95	0.0	0.0	0.21135	0.18494	0.24270	0.0	0.0	0.17818	0.0	0.18284

- It is more robust and reliable as it always generates efficient solutions and is able to produce both unbalanced and balanced compromise solutions based on the investor preferences.
- The obtained solutions are consistent with the investor preferences, i.e., the consistency between importance weights of objectives (γ_1, γ_2) and satisfaction degrees $(\mu_{f_1}(x), \mu_{f_2}(x))$.
- It is more flexible because it provides different efficient solutions for a specific problem instance with a given importance weight vector of objectives (γ_1, γ_2) by varying values of the compensation coefficient (w).

5.5 Comments

In this chapter, we have presented the following facts:

- A hybrid bi-objective credibility-based fuzzy mathematical programming model for portfolio selection problem under fuzzy environment has been discussed.
- To deal with imprecise parameters, a hybrid credibility-based approach that combines the expected value and chance constrained programming techniques has been presented.
- The hybrid bi-objective fuzzy optimization model has been solved by using a two-phase approach.
- Using the computational results, it has been shown that the fuzzy interactive approach is a very promising approach that can provide both unbalanced and balanced efficient solutions based on the investor preferences. Moreover, it also offers appropriate flexibility to generate different portfolios in order to help the investor in selecting a preferred compromise portfolio.

Chapter 6
Multi-criteria Fuzzy Portfolio Optimization

Abstract. In this chapter, we describe fuzzy portfolio selection models using five criteria: short term return, long term return, dividend, risk and liquidity. For portfolio return, we consider short term return (average performance of the asset during a 12-month period), long term return (average performance of the asset during a 36-month period) and annual dividend. This is done in order to capture subjective preferences of the investors for portfolio return. For a given expected return, the negative semi-absolute deviation is penalized which quantifies portfolio risk. Further, we categorize all individual investor attitudes towards bearing portfolio risk into one of the following two distinct classes: aggressive (weak risk aversion attitude) and conservative (strong risk aversion attitude). The nonlinear S-shape membership functions are employed to express vague aspiration levels of the investor regarding the multiple criteria used for portfolio selection.

6.1 Multi-criteria Portfolio Selection Model

Most of the existing portfolio selection models consider return and risk as the two fundamental factors that govern investors' choice. However, it is often found that not all the relevant information for portfolio selection can be captured in terms of return and risk only. The other considerations/criteria might be of equal, if not greater, importance to investors. By considering these in the portfolio selection model, it may be possible to obtain portfolios in which a deficit on account of the return and risk criteria is more than compensated by portfolio performance on other criteria, resulting in greater overall satisfaction for investors. Thus, multiple criteria portfolio selection models have received great attention in recent past. Some of the relevant references in this direction include Arenas et al. [3], Ehrgott et al. [30], Fang et al. [33], Gupta et al. [41, 42, 43, 44], Li et al. [82], Nanda et al. [96].

Here, we formulate portfolio selection problem as an optimization problem with multiple objectives assuming that the investor allocate his/her wealth

P. Gupta et al., *Fuzzy Portfolio Optimization*,
Studies in Fuzziness and Soft Computing 316,
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among n assets that offer random rates of return. We introduce some notation as follows:

6.1.1 Notation

r_i: the expected rate of return of the i-th asset,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

d_i: the annual dividend of the i-th asset,

r_i^{12}: the average performance of the i-th asset during a 12-month period,

r_i^{36}: the average performance of the i-th asset during a 36-month period,

r_{it}: the historical return of the i-th asset over the past period t,

u_i: the maximal fraction of the capital allocated to the i-th asset,

l_i: the minimal fraction of the capital allocated to the i-th asset,

L: the minimum desired level of portfolio liquidity,

$\tilde{L}_i$: the fuzzy turnover rate of the i-th asset,

h: the number of assets held in the portfolio,

T: the total time span.

We consider the following objective functions and constraints in the multiobjective portfolio selection problem.

6.1.2 Objective Functions

Short Term Return

The short term return of the portfolio is expressed as

$$f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i,$$

where $r_i^{12} = \frac{1}{12}\sum_{t=1}^{12} r_{it}$, $i = 1, 2, \ldots, n$; r_{it} is determined from the historical data.

Long Term Return
The long term return of the portfolio is expressed as

$$f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i \, ,$$

where $r_i^{36} = \dfrac{1}{36}\displaystyle\sum_{t=1}^{36} r_{it} \, , \; i = 1, 2, \ldots, n.$

Annual Dividend
The annual dividend of the portfolio is expressed as

$$f_3(x) = \sum_{i=1}^{n} d_i x_i \, .$$

Risk
The expected semi-absolute deviation of the portfolio return below the expected return is expressed as

$$f_4(x) = w(x) = \frac{1}{T}\sum_{t=1}^{T} w_t(x) = \sum_{t=1}^{T} \frac{\left|\displaystyle\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \displaystyle\sum_{i=1}^{n}(r_i - r_{it})x_i}{2T} \, .$$

6.1.3 Constraints

Liquidity
We assume that turnover rates used to determine liquidity of the assets are vague estimates and conform to trapezoidal possibility distribution. A trapezoidal fuzzy number $\tilde{A} = (a, b, \alpha, \beta)$ with tolerance interval $[a, b]$, left width α and right width β, has the following membership function

$$\mu_{\tilde{A}}(t') = \begin{cases} \dfrac{t' - (a - \alpha)}{\alpha} \, , & \text{if } \; a - \alpha \le t' \le a, \\ 1 \, , & \text{if } \; a \le t' \le b, \\ \dfrac{b + \beta - t'}{\beta} \, , & \text{if } \; b \le t' \le b + \beta, \\ 0 \, , & \text{otherwise.} \end{cases}$$

Let the trapezoidal fuzzy number $\tilde{L}_i = (L_{a_i}, L_{b_i}, L_{\alpha_i}, L_{\beta_i})$ denotes the turnover rate of the i-th asset. Then, the turnover rate of the portfolio is expressed as $\displaystyle\sum_{i=1}^{n} \tilde{L}_i x_i$. Using fuzzy extension principle [125], the crisp possibilistic mean value of fuzzy turnover rate of the i-th asset is given by

$$E(\tilde{L}_i) = \int_0^1 \gamma(L_{a_i} - (1-\gamma)L_{\alpha_i} + L_{b_i} + (1-\gamma)L_{\beta_i})d\gamma$$
$$= \frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6}\,.$$

Therefore, the crisp possibilistic mean value of the portfolio liquidity is obtained as

$$E(\tilde{L}(x)) = E\left(\sum_{i=1}^{n} \tilde{L}_i x_i\right) = \sum_{i=1}^{n}\left(\frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6}\right)x_i\,.$$

Further, to maintain portfolio liquidity at a specified level L given by the investor, we use the following constraint.

$$\sum_{i=1}^{n}\left(\frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6}\right)x_i \geq L\,.$$

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1\,.$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,.$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,.$$

Number of assets held in the portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h\,.$$

No short selling of assets is expressed as

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

6.1.4 The Decision Problem

The constrained multiobjective portfolio selection problem is formulated as

$$\textbf{P(6.1)} \quad \max \ f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i$$

$$\max \ f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i$$

$$\max \ f_3(x) = \sum_{i=1}^{n} d_i x_i$$

$$\min \ f_4(x) = w(x) = \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2T}$$

subject to

$$\sum_{i=1}^{n}\left(\frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6}\right)x_i \geq L\,,$$

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$\sum_{i=1}^{n} y_i = h\,,$$

$$x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,,$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,,$$

$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,.$$

Note that portfolio liquidity can also be considered as one of the objective functions in the problem P(6.1). To eliminate the absolute-valued function in problem P(6.1), we transform it into the following multiobjective mixed integer linear programming problem.

$$
\begin{aligned}
\mathbf{P(6.2)}\quad & \max\ f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i \\
& \max\ f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i \\
& \max\ f_3(x) = \sum_{i=1}^{n} d_i x_i \\
& \min\ f_4(p) = w(p) = \frac{1}{T}\sum_{t=1}^{T} p_t \\
& \text{subject to} \\
& \sum_{i=1}^{n}\left(\frac{L_{a_i}+L_{b_i}}{2} + \frac{L_{\beta_i}-L_{\alpha_i}}{6}\right)x_i \geq L\,, \\
& p_t + \sum_{i=1}^{n}(r_{it}-r_i)x_i \geq 0, \quad t = 1,2,\ldots,T\,, \\
& \sum_{i=1}^{n} x_i = 1\,, \\
& \sum_{i=1}^{n} y_i = h\,, \\
& x_i \leq u_i y_i\,, \quad i = 1,2,\ldots,n\,, \\
& x_i \geq l_i y_i\,, \quad i = 1,2,\ldots,n\,, \\
& x_i \geq 0\,, \quad i = 1,2,\ldots,n\,, \\
& p_t \geq 0\,, \quad t = 1,2,\ldots,T\,, \\
& y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,n\,.
\end{aligned}
$$

6.2 Fuzzy Multi-criteria Portfolio Selection Models and Solution Methodology

We discuss a fuzzy multiobjective portfolio selection problem based on vague aspiration levels of the investor to determine a satisfying portfolio selection strategy. The investor indicate aspiration levels on the basis of his/her prior experience and knowledge. We use a nonlinear S-shape membership function to express vague aspiration levels of the investor. The S-shape membership function is defined as

$$
f(x) = \frac{1}{1+\exp(-\alpha x)}
$$

where α, $0 < \alpha < \infty$ is a fuzzy parameter which measures the degree of vagueness. Note that a logistic function is considered an appropriate function to represent vague aspiration levels of the investor in financial decision making problems.

Remark 6.1. *The logistic (S-shape) membership function has shape similar to that of tangent hyperbolic function, but it is more easily handled than the tangent hyperbolic function. Further, the logistic membership function preserves linearity even when the operator 'product' is used instead of the operator 'min' for aggregating the overall satisfaction to arrive at the fuzzy set decision. Moreover, a trapezoidal membership function is an approximation to a logistic function.*

Let us consider that the four objectives (short term return, long term return, annual dividend and risk) and the constraint on the liquidity of the portfolio are vague and uncertain. We define the vague aspiration levels of the investor as follows:

- The membership function of the goal of expected short term return is given by

$$\mu_{r^{12}}(x) = \frac{1}{1 + \exp\left(-\alpha_{r^{12}}\left(\sum_{i=1}^{n} r_i^{12} x_i - r_m^{12}\right)\right)},$$

  where r_m^{12} is the mid-point (middle aspiration level for the expected short term return) at which the membership function value is 0.5 and $\alpha_{r^{12}}$ is provided by the investor based on his/her degree of satisfaction of the goal (see Fig. 6.1).

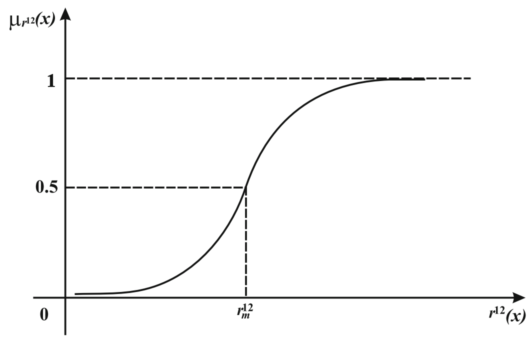

Fig. 6.1 Membership function of the goal of expected short term return

- The membership function of the goal of expected long term return is given by

$$\mu_{r^{36}}(x) = \frac{1}{1 + \exp\left(-\alpha_{r^{36}}\left(\sum_{i=1}^{n} r_i^{36} x_i - r_m^{36}\right)\right)},$$

where r_m^{36} is the mid-point (middle aspiration level for the expected long term return) at which the membership function value is 0.5 and $\alpha_{r^{36}}$ is provided by the investor based on his/her degree of satisfaction of the goal (see Fig. 6.2).

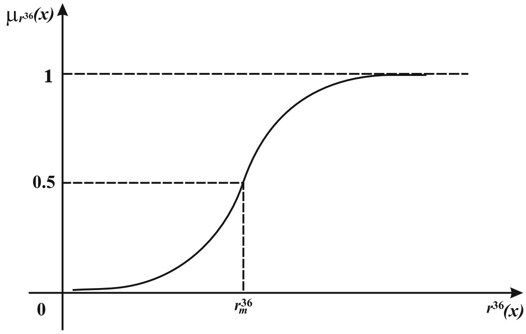

Fig. 6.2 Membership function of the goal of expected long term return

- The membership function of the goal of annual dividend is given by

$$\mu_d(x) = \frac{1}{1 + \exp\left(-\alpha_d\left(\sum_{i=1}^{n} d_i x_i - d_m\right)\right)},$$

where d_m is the mid-point (middle aspiration level for the annual dividend) where the membership function value is 0.5 and α_d is provided by the investor based on his/her degree of satisfaction of the goal (see Fig. 6.3).
- The membership function of the goal of risk is given by

$$\mu_w(x) = \frac{1}{1 + \exp\left(\alpha_w(w(x) - w_m)\right)},$$

where w_m is the mid-point (middle aspiration level for the portfolio risk) at which the membership function value is 0.5 and α_w is provided by the investor based on his/her degree of satisfaction of the goal (see Fig. 6.4).

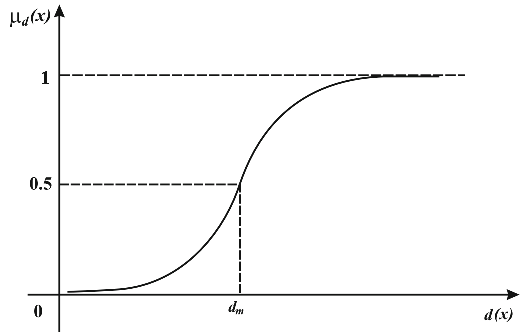

Fig. 6.3 Membership function of the goal of annual dividend

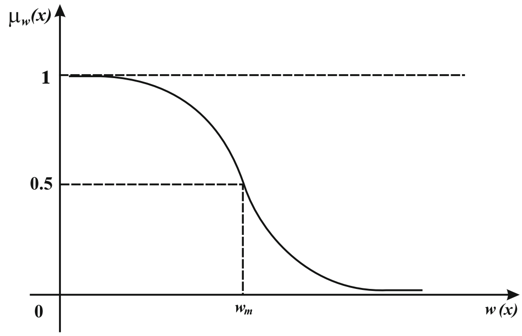

Fig. 6.4 Membership function of the goal of risk

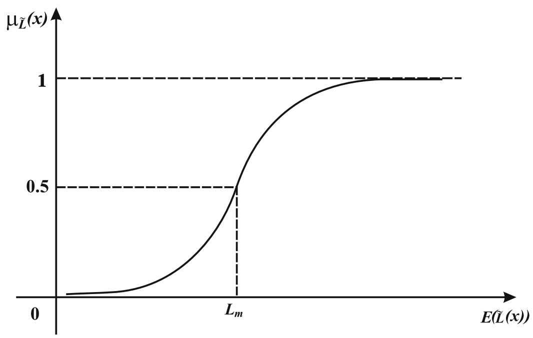

Fig. 6.5 Membership function of the liquidity constraint

- The membership function of the portfolio liquidity threshold is given by

$$\mu_{\tilde{L}}(x) = \frac{1}{1 + \exp\left(-\alpha_L (E(\tilde{L}(x)) - L_m)\right)},$$

where L_m is the mid-point (middle aspiration level for the liquidity threshold) at which the membership function value is 0.5 and α_L is provided by the investor based on his/her degree of satisfaction (see Fig. 6.5).

Remark 6.2. *The shape parameters* $\alpha_{r^{12}}$, $\alpha_{r^{36}}$, α_d, α_w *and* α_L *determine shapes of the membership functions* $\mu_{r^{12}}(x)$, $\mu_{r^{36}}(x)$, $\mu_d(x)$, $\mu_w(x)$ *and* $\mu_{\tilde{L}}(x)$ *respectively, and are selected in the range* $(0, \infty)$*. The larger these parameters get, the less their vagueness becomes.*

Remark 6.3. *The mid-points* r_m^{12}, r_m^{36}, d_m, w_m *and* L_m *are determined by taking* $r_m^{12} = \dfrac{r_N^{12} + r_S^{12}}{2}$, $r_m^{36} = \dfrac{r_N^{36} + r_S^{36}}{2}$, $d_m = \dfrac{d_N + d_S}{2}$, $w_m = \dfrac{w_N + w_S}{2}$, $L_m = \dfrac{L_N + L_S}{2}$*. Here,* r_N^{12}, r_N^{36}, d_N, w_N *and* L_N *are the necessity levels and* r_S^{12}, r_S^{36}, d_S, w_S *and* L_S *are the sufficiency levels indicated by the investor. Note that linear membership functions such as triangular and trapezoidal functions show a necessity level and a sufficiency level at 0 and 1, respectively. On the other hand, a necessity level and/or a sufficiency level may be approximated for S-shape membership functions.*

Now, using Bellman-Zadeh's maximization principle [7], the fuzzy multiobjective portfolio selection problem is formulated as follows:

P(6.3) $\max\ \eta$

subject to

$$\eta \le \mu_{r^{12}}(x)\,, \tag{6.1}$$
$$\eta \le \mu_{r^{36}}(x)\,, \tag{6.2}$$
$$\eta \le \mu_d(x)\,, \tag{6.3}$$
$$\eta \le \mu_w(x)\,, \tag{6.4}$$
$$\eta \le \mu_{\tilde{L}}(x)\,, \tag{6.5}$$
$$\sum_{i=1}^{n} x_i = 1\,, \tag{6.6}$$
$$\sum_{i=1}^{n} y_i = h\,, \tag{6.7}$$
$$x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n, \tag{6.8}$$
$$x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n, \tag{6.9}$$
$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,, \tag{6.10}$$
$$x_i \ge 0\,, \quad i = 1, 2, \ldots, n, \tag{6.11}$$
$$0 \le \eta \le 1\,. \tag{6.12}$$

The problem P(6.3) is a mixed integer nonlinear programming problem. We transform it into a mixed integer linear programming problem. Let the constraints involving exponential function in the problem P(6.3) be rewritten as the following inequations:

$$\alpha_{r^{12}}\left(\sum_{i=1}^{n} r_i^{12}x_i - r_m^{12}\right) \geq \log\frac{\eta}{1-\eta},$$

$$\alpha_{r^{36}}\left(\sum_{i=1}^{n} r_i^{36}x_i - r_m^{36}\right) \geq \log\frac{\eta}{1-\eta},$$

$$\alpha_{d}\left(\sum_{i=1}^{n} d_ix_i - d_m\right) \geq \log\frac{\eta}{1-\eta},$$

$$-\alpha_w(w(x) - w_m) \geq \log\frac{\eta}{1-\eta},$$

$$\alpha_L(E(\tilde{L}(x)) - L_m) \geq \log\frac{\eta}{1-\eta}.$$

Further, let $\theta = \log\frac{\eta}{1-\eta}$, so that $\eta = \frac{1}{1+\exp(-\theta)}$; hence, maximizing η also maximizes θ. Thus, the problem P(6.3) is transformed into the following equivalent mixed integer linear programming problem.

$$\begin{aligned}
\textbf{P(6.4)}\quad & \max\ \theta \\
& \text{subject to} \\
& \theta \leq \alpha_{r^{12}}\left(\sum_{i=1}^{n} r_i^{12}x_i - r_m^{12}\right), \\
& \theta \leq \alpha_{r^{36}}\left(\sum_{i=1}^{n} r_i^{36}x_i - r_m^{36}\right), \\
& \theta \leq \alpha_{d}\left(\sum_{i=1}^{n} d_ix_i - d_m\right), \\
& \theta \leq \alpha_w\left(w_m - w(x)\right), \\
& \theta \leq \alpha_L\left(\sum_{i=1}^{n}\left(\frac{L_{a_i}+L_{b_i}}{2} + \frac{L_{\beta_i}-L_{\alpha_i}}{6}\right)x_i - L_m\right), \\
& \text{and Constraints (6.6)} - \text{(6.11)}.
\end{aligned}$$

Note that $\theta \in]-\infty, +\infty[$. The absolute-valued function in the expression of $w(x)$ can be eliminated on the same lines as discussed in Section 6.1.4. The fuzzy portfolio selection problem P(6.3)/P(6.4) leads to a fuzzy decision that simultaneously satisfies all the fuzzy objectives. Then, we determine the maximizing decision as the maximum degree of membership for the fuzzy decision. In this approach, the relationship between various objectives in a fuzzy environment is considered fully symmetric, i.e., all fuzzy objectives are treated equivalent. This approach is efficient in computation but it may provide ‘uniform’ membership degrees for all fuzzy objectives even when achievement of some objective(s) is more stringently required. To incorporate relative importance of various fuzzy objectives/criteria in portfolio selection problem, we use a ‘weighted additive model’ [116]. The weighted additive model of the fuzzy multiobjective portfolio selection problem is formulated as follows:

$$\begin{aligned} \mathbf{P(6.5)}\quad & \max \sum_{p=1}^{5} \omega_p \eta_p \\ & \text{subject to} \\ & \eta_1 \le \mu_{r^{12}}(x)\,, \\ & \eta_2 \le \mu_{r^{36}}(x)\,, \\ & \eta_3 \le \mu_d(x)\,, \\ & \eta_4 \le \mu_w(x)\,, \\ & \eta_5 \le \mu_{\tilde{L}}(x)\,, \\ & 0 \le \eta_p \le 1\,, \quad p = 1,2,\ldots,5, \\ & \text{and Constraints } (6.6) - (6.11)\,, \end{aligned}$$

where ω_p is the relative weight of the p-th objective given by the investor such that $\omega_p > 0$ and $\sum_{p=1}^{5} \omega_p = 1$.

The max-min approach used in the formulation of the problems P(6.3)/P(6.4) and P(6.5) possesses good computational properties; however, the efficiency of the solution is not guaranteed. The compromise approach [120] and the two-phase approach [81] have been proposed in literature to treat inefficiency. Using compromise approach, the decision maker can choose explicitly a desirable achievement degree for each fuzzy objective function by adding additional constraints $\eta_p \ge \beta_p$, where β_p is the desirable achievement degree for the p-th fuzzy objective function. Note that if we increase the desirable achievement degree for an objective function then the value of that objective function is more close to the optimal value but it may yield other objective function values far from the optimal values because of multiobjective nature of the problem. As a result, when a decision maker requires a very high desirable achievement degree for each fuzzy objective function, we may end up with ‘no feasible solution’; thus, a compromise between objective functions should be made. To overcome the difficulty in selecting proper desirable achievement degree for each fuzzy objective function, we use a two-phase approach [81] in which the desirable achievement degree is taken as the degree of satisfaction corresponding to the solution from the max-min approach. Consequently, the problems P(6.6) and P(6.7) are solved corresponding to the problems P(6.4) and P(6.5), respectively, in the second-phase to ensure efficiency of the solutions obtained.

$$\textbf{P(6.6)} \quad \max \sum_{p=1}^{5} \omega_p \theta_p$$

subject to

$$\log \frac{\mu_{r^{12}}(x^*)}{1-\mu_{r^{12}}(x^*)} \le \theta_1 \le \alpha_{r^{12}} \left(\sum_{i=1}^{n} r_i^{12} x_i - r_m^{12} \right),$$

$$\log \frac{\mu_{r^{36}}(x^*)}{1-\mu_{r^{36}}(x^*)} \le \theta_2 \le \alpha_{r^{36}} \left(\sum_{i=1}^{n} r_i^{36} x_i - r_m^{36} \right),$$

$$\log \frac{\mu_{d}(x^*)}{1-\mu_{d}(x^*)} \le \theta_3 \le \alpha_{d} \left(\sum_{i=1}^{n} d_i x_i - d_m \right),$$

$$\log \frac{\mu_{w}(x^*)}{1-\mu_{w}(x^*)} \le \theta_4 \le \alpha_{w} \left(w_m - w(x) \right),$$

$$\log \frac{\mu_{\tilde{L}}(x^*)}{1-\mu_{\tilde{L}}(x^*)} \le \theta_5 \le \alpha_{L} \left(\sum_{i=1}^{n} \left(\frac{L_{a_i}+L_{b_i}}{2} + \frac{L_{\beta_i}-L_{\alpha_i}}{6} \right) x_i - L_m \right),$$

and Constraints (6.6) – (6.11),

where x^* is an optimal solution of P(6.4), $\omega_1 = \ldots = \omega_5$, $\omega_p > 0$, $\sum_{p=1}^{5} \omega_p = 1$ and $\theta_p \in]-\infty, +\infty[\,,\ p = 1, 2, \ldots, 5$.

$$\textbf{P(6.7)} \quad \max \sum_{p=1}^{5} \omega_p \eta_p$$

subject to

$$\mu_{r^{12}}(x^{**}) \le \eta_1 \le \mu_{r^{12}}(x),$$
$$\mu_{r^{36}}(x^{**}) \le \eta_2 \le \mu_{r^{36}}(x),$$
$$\mu_{d}(x^{**}) \le \eta_3 \le \mu_{d}(x),$$
$$\mu_{w}(x^{**}) \le \eta_4 \le \mu_{w}(x),$$
$$\mu_{\tilde{L}}(x^{**}) \le \eta_5 \le \mu_{\tilde{L}}(x),$$
$$0 \le \eta_p \le 1, \quad p = 1, 2, \ldots, 5,$$

and Constraints (6.6) – (6.11),

where x^{**} is an optimal solution of P(6.5), ω_p is the relative weight of the p-th objective given by the investor such that $\omega_p > 0$ and $\sum_{p=1}^{5} \omega_p = 1$.

6.3 Numerical Illustration

To demonstrate the usefulness of the fuzzy optimization models for the portfolio selection, a real-world empirical study is presented for an imaginary investor on the data set extracted from NSE, Mumbai, India in respect of randomly selected 20 assets. Table 6.1 provides the data corresponding to expected short term return, expected long term return and risk.

Table 6.1 Input data of assets corresponding expected short term return, expected long term return and risk

Company	Expected short term return	Expected long term return	Risk
A B B Ltd. (ABB)	0.06139	0.06252	0.04241
ACC Ltd. (ACC)	0.07012	0.05214	0.03732
Allahabad Bank (AHB)	-0.00073	0.04638	0.05204
Ashok Leyland Ltd. (ALL)	0.05507	0.01758	0.06471
Bajaj Auto Ltd. (BAL)	0.07276	0.05144	0.03748
Bharat Electronics Ltd. (BEL)	0.03667	0.04443	0.04715
Bharat Heavy Electricals Ltd. (BHE)	0.07479	0.06472	0.04890
Bharat Petroleum Corpn. Ltd. (BPC)	0.00438	0.01840	0.0432
Cipla Ltd. (CIL)	0.01435	0.01128	0.06651
Dr. Reddy's Laboratories Ltd. (DRL)	0.05630	0.01672	0.04128
Hindustan Motors (HIM)	0.02317	0.02123	0.04166
Infosys Technologies Ltd. (ITL)	-0.04958	0.00948	0.04913
I T C Ltd. (ITC)	0.03368	0.02278	0.05161
Mahanagar Tele. Nig. Ltd. (MTN)	0.08718	0.08475	0.05041
Siemens Ltd. (SIL)	0.02667	0.04690	0.05527
Tata Power Co. Ltd. (TPC)	0.08307	0.07791	0.06113
Titan Industries Ltd (TIL)	0.13598	0.09203	0.05357
Voltas Ltd. (VOL)	0.07433	0.05407	0.05094
Videsh Sanchar Nigam Ltd. (VSN)	0.03422	0.04784	0.06062
Wipro Ltd. (WIL)	-0.01825	0.00470	0.05418

Table 6.2 The fuzzy turnover rates of the assets

Assets	ABB	ACC	AHB	ALL
$\tilde{L}$	(0.0008, 0.0032, 0.0004,0.0018)	(0.004,0.016,0.002,0.004)	(0.004,0.016, 0.002, 0.014)	(0.003, 0.009, 0.001,0.004)
Assets	BAL	BEL	BHE	BPC
$\tilde{L}$	(0.0012,0.0024, 0.0003, 0.0005)	(0.0024, 0.0084,0.0010, 0.0020)	(0.0020, 0.008, 0.0010, 0.0010)	(0.0024, 0.0096, 0.0010, 0.0020)
Assets	CIL	DRL	HIM	ITL
$\tilde{L}$	(0.0018,0.0038,0.0010, 0.0015)	(0.0019, 0.0039,0.0006, 0.0006)	(0.0030, 0.009, 0.0010, 0.0020)	(0.0006, 0.0018, 0.0002, 0.0002)
Assets	ITC	MTN	SIL	TPC
$\tilde{L}$	(0.004, 0.016, 0.0023, 0.004)	(0.0007, 0.0031, 0.0004,0.0010)	(0.0031, 0.0121, 0.0014, 0.0032)	(0.005, 0.034, 0.002, 0.016)
Assets	TIL	VOL	VSN	WIL
$\tilde{L}$	(0.0011, 0.0041, 0.0005, 0.0019)	(0.003, 0.015,0.0015, 0.009)	(0.006, 0.03, 0.003, 0.01)	(0.0027, 0.0117, 0.0007,0.0060)

Table 6.2 provides liquidity profile of all the assets using trapezoidal fuzzy numbers. The liquidity profile is based on the daily turnover rate for each of the assets. Since we have assumed that the turnover rates are trapezoidal fuzzy numbers, we need to estimate the tolerance interval, the left spread and the right spread of the fuzzy numbers. In the real-world applications of portfolio selection, the values of these parameters can be obtained by using the Delphi Method [84]. As an illustration, consider the calculation of fuzzy turnover rate of asset ABB. First, we use historical data (daily turnover rates from April 1, 2005 to March 31, 2008) to calculate frequency of historical turnover rates. We find that most of the historical turnover rates fall into the intervals [0.0004, 0.0012], [0.0012, 0.0020], [0.0020, 0.0028] and [0.0028, 0.0036]. We take the mid-points of the intervals [0.0004, 0.0012] and [0.0028, 0.0036] as the left and the right-end points of the tolerance interval, respectively. Thus, the tolerance interval of the fuzzy turnover rate become [0.0008, 0.0032]. By observing all the historical data, we use 0.0004 and 0.005 as the minimum possible value and the maximum possible value, respectively, of the uncertain turnover rate. Thus, the left spread is 0.0004 and the right spread is 0.0018. The fuzzy turnover rate of asset ABB hence becomes [0.0008, 0.0032, 0.0004, 0.0018] and the corresponding crisp possibilistic mean value is 0.0022. Similarly, we obtain fuzzy turnover rates and the corresponding crisp possibilistic mean values of all 20 assets. Table 6.3 provides crisp possibilistic mean values of all 20 assets.

Table 6.3 Crisp possibilistic mean values of all 20 assets (liquidity)

ABB	ACC	AHB	ALL	BAL	BEL	BHE	BPC	CIL	DRL
0.0022	0.0103	0.0120	0.0065	0.0019	0.0056	0.0050	0.0062	0.0029	0.0029
HIM	ITL	ITC	MTN	SIL	TPC	TIL	VOL	VSN	WIL
0.0062	0.0103	0.0012	0.0020	0.0079	0.0218	0.0028	0.0103	0.0192	0.0081

The input parameters of the problem instances solved are summarized in Table 6.4. The 20 financial assets form the population from which we construct a portfolio comprising 8 assets with the corresponding upper and lower bounds of capital budget allocation. The objective is to maximize the degree of satisfaction in regard to maximization of short term and long term portfolio returns, portfolio liquidity and minimization of portfolio risk. Here, we ignore the issue of annual portfolio dividend with a view to focus on the portfolio returns arising from the price movements of the underlying assets (return on a asset is represented by the percentage change in daily closing prices).

Table 6.4 The input parameters of the problem instances

	Model P(6.4) for aggressive investor	Model P(6.4) for conservative investor	Model P(6.5)
No. of assets	20	20	20
No. of criteria	4	4	4
Membership functions	nonlinear S-shape	nonlinear S-shape	nonlinear S-shape
Shape parameters	(i) $\alpha_{r12} = 600$, $\alpha_{r36} = 600$, $\alpha_w = 800$, $\alpha_L = 600$, (ii) $\alpha_{r12} = 500$, $\alpha_{r36} = 500$, $\alpha_w = 1000$, $\alpha_L = 500$, (iii) $\alpha_{r12} = 400$, $\alpha_{r36} = 400$, $\alpha_w = 1200$, $\alpha_L = 400$,	(i) $\alpha_{r12} = 600$, $\alpha_{r36} = 600$, $\alpha_w = 800$, $\alpha_L = 600$, (ii) $\alpha_{r12} = 500$, $\alpha_{r36} = 500$, $\alpha_w = 1000$, $\alpha_L = 500$, (iii) $\alpha_{r12} = 400$, $\alpha_{r36} = 400$, $\alpha_w = 1200$, $\alpha_L = 400$,	$\alpha_{r12} = 600$, $\alpha_{r36} = 600$, $\alpha_w = 800$, $\alpha_L = 600$,
Middle aspiration levels	$r_m^{12} = 0.05$, $r_m^{36} = 0.0475$ $w_m = 0.0525$, $L_m = 0.0125$	$r_m^{12} = 0.045$, $r_m^{36} = 0.042$ $w_m = 0.05$, $L_m = 0.01$	(i) $r_m^{12} = 0.05$, $r_m^{36} = 0.0475$ $w_m = 0.0525$, $L_m = 0.0125$ (ii) $r_m^{12} = 0.112$, $r_m^{36} = 0.075$ $w_m = 0.0525$, $L_m = 0.0083$ (iii) $r_m^{12} = 0.1$, $r_m^{36} = 0.075$ $w_m = 0.0625$, $L_m = 0.0125$
No. of assets held in the portfolio	8	8	8
Criteria weights	—	—	(i) $\omega_1 = 0.4$, $\omega_2 = 0.35$ $\omega_3 = 0.15$, $\omega_4 = 0.1$ (ii) $\omega_1 = 0.2$, $\omega_2 = 0.4$ $\omega_3 = 0.35$, $\omega_4 = 0.05$ (iii) $\omega_1 = 0.2$, $\omega_2 = 0.3$ $\omega_3 = 0.35$, $\omega_4 = 0.15$

In what follows, we present computational results by taking hypothetical situations representing aggressive and conservative portfolio selection strategies. The investor pursuing aggressive strategy aspire for higher returns and liquidity even though it may result in higher risk. Conversely, the investor pursuing conservative strategy prefer lower risk even though such a strategy may result in lower returns and liquidity. The comparative values of the aspiration levels in Table 6.4 amplify diversity of the investor behavior. All the optimization models are coded and solved using LINGO 12.0.

• **Aggressive portfolio selection strategy**

To obtain a portfolio selection for the investor that has an aggressive and optimistic mind, we formulate and solve the model P(6.4) using the input data from Tables 6.1, 6.3-6.4, $r_m^{12} = 0.05$, $r_m^{36} = 0.0475$, $w_m = 0.0525$, $L_m = 0.0125$, $h = 8$, $l_i = 0.01$ and $u_i = 0.55$, $i = 1, 2, \ldots, 20$.

$$
\begin{aligned}
&\max\ \theta \\
&\text{subject to} \\
&36.834x_1 + 42.072x_2 - 0.438x_3 + 33.042x_4 + 43.656x_5 + 22.002x_6 \\
&+44.874x_7 + 2.628x_8 + 8.61x_9 + 33.78x_{10} + 13.902x_{11} - 29.748x_{12} \\
&+20.208x_{13} + 52.308x_{14} + 16.002x_{15} + 49.842x_{16} + 81.588x_{17} + 44.598x_{18} \\
&+20.532x_{19} - 10.95x_{20} - 30 \geq \theta\,, \\
&37.512x_1 + 31.284x_2 + 27.828x_3 + 10.548x_4 + 30.864x_5 + 26.658x_6 \\
&+38.832x_7 + 11.04x_8 + 6.798x_9 + 10.032x_{10} + 12.738x_{11} + 5.688x_{12} \\
&+13.688x_{13} + 50.85x_{14} + 28.14x_{15} + 46.746x_{16} + 55.218x_{17} + 32.442x_{18} \\
&+28.704x_{19} + 2.82x_{20} - 28.5 \geq \theta\,, \\
&-33.928x_1 - 29.856x_2 - 41.632x_3 - 51.768x_4 - 29.984x_5 - 37.72x_6 \\
&-39.12x_7 - 34.56x_8 - 53.208x_9 - 33.024x_{10} - 33.328x_{11} - 39.304x_{12} \\
&-41.288x_{13} - 40.328x_{14} - 44.216x_{15} - 48.904x_{16} - 42.856x_{17} - 40.752x_{18} \\
&-48.496x_{19} - 43.344x_{20} + 42 \geq \theta\,, \\
&1.32x_1 + 6.18x_2 + 7.2x_3 + 3.9x_4 + 1.14x_5 + 3.36x_6 + 3x_7 + 3.72x_8 + 1.74x_9 \\
&+1.74x_{10} + 3.72x_{11} + 0.72x_{12} + 6.18x_{13} + 1.2x_{14} + 4.74x_{15} + 13.08x_{16} \\
&+1.68x_{17} + 6.18x_{18} + 11.52x_{19} + 4.86x_{20} - 7.5 \geq \theta\,, \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \\
&+x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,, \\
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} \\
&+y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,, \\
&x_i - 0.01y_i \geq 0\,, \quad i = 1,2,\ldots,20\,, \\
&x_i - 0.55y_i \leq 0\,, \quad i = 1,2,\ldots,20\,, \\
&y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,20\,, \\
&x_i \geq 0\,, \quad i = 1,2,\ldots,20\,.
\end{aligned}
$$

If the investor is not satisfied with the portfolio obtained, more portfolios can be generated by varying values of the shape parameters $\alpha_{r^{12}}$, $\alpha_{r^{36}}$, α_w and α_L in the above problem. Corresponding to three different sets of values of these parameters, the computational results are summarized in Table 6.5. Table 6.6 presents proportions of the assets in the obtained portfolios. To check efficiency of the solutions obtained, the two-phase approach is applied in which the problem P(6.6) is solved. We find that the solution obtained corresponding to the shape parameters $\alpha_{r^{36}} = 500, \alpha_{r^{12}} = 500, \alpha_w = 1000, \alpha_L = 500$ is not efficient. In this case, the recourse to the two-phase approach produces the efficient solution listed in Tables 6.7-6.8. The solution criteria vector $(0.0508, 0.0645, 0.0743, 0.0160)$ of the Table 6.7 dominates the solution criteria vector $(0.0508, 0.0628, 0.0710, 0.0160)$ of the Table 6.5 corresponding to the shape parameters $\alpha_{r^{36}} = 500, \alpha_{r^{12}} = 500$, $\alpha_w = 1000, \alpha_L = 500$.

Table 6.5 Summary results of portfolio selection for aggressive investor

Shape parameters & variables						Risk	Expected return		Liquidity
η	θ	α_{r36}	α_{r12}	α_w	α_L		Long term	Short term	
0.8707	1.9072	600	600	800	600	0.0501	0.0621	0.0707	0.0157
0.8512	1.744	500	500	1000	500	0.0508	0.0628	0.0710	0.0160
0.8164	1.4922	400	400	1200	400	0.0513	0.0633	0.0713	0.0162

Table 6.6 The proportions of the assets in the obtained portfolios for aggressive investor

Shape parameters				Allocation				
$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	ABB	ACC	AHB	ALL	BAL
600	600	800	600	0	0.4	0.02	0	0.02
500	500	1000	500	0	0.374	0.02	0	0.019
400	400	1200	400	0	0.353	0.02	0	0.019
				BEL	BHE	BPC	CIL	DRL
				0	0	0.028	0	0
				0	0	0.028	0	0
				0	0	0.028	0	0
				HIM	ITL	ITC	MTN	SIL
				0	0	0.023	0	0
				0	0	0.023	0	0
				0	0	0.023	0	0
				TPC	TIL	VOL	VSN	WIL
				0.484	0	0	0.01	0.015
				0.511	0	0	0.01	0.015
				0.532	0	0	0.01	0.015

Table 6.7 Summary result of portfolio selection (improved solution)

$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	Risk	Expected return		Liquidity
					Long term	Short term	
500	500	1000	500	0.0508	0.0645	0.0743	0.0160

- **Conservative portfolio selection strategy**

To obtain a portfolio selection for the investor that has an conservative and pessimistic mind, we formulate and solve the model P(6.4) using the input data from Tables 6.1, 6.3-6.4, $r_m^{12} = 0.045$, $r_m^{36} = 0.042$, $w_m = 0.05$ and $L_m = 0.01$, $h = 8$, $l_i = 0.01$, and $u_i = 0.55$, $i = 1, 2, \ldots, 20$.

Table 6.8 The proportions of the assets in the obtained portfolio corresponding to improved solution

	Allocation									
	ABB	ACC	AHB	ALL	BAL	BEL	BHE	BPC	CIL	DRL
Portfolio	0	0.374	0.02	0	0.019	0	0.02	0	0	0
	HIM	ITL	ITC	MTN	SIL	TPC	TIL	VOL	VSN	WIL
	0	0	0.023	0	0	0.502	0	0.032	0.01	0

$$
\begin{aligned}
&\max\ \theta \\
&\text{subject to} \\
&36.834x_1 + 42.072x_2 - 0.438x_3 + 33.042x_4 + 43.656x_5 + 22.002x_6 \\
&+44.874x_7 + 2.628x_8 + 8.61x_9 + 33.78x_{10} + 13.902x_{11} - 29.748x_{12} \\
&+20.208x_{13} + 52.308x_{14} + 16.002x_{15} + 49.842x_{16} + 81.588x_{17} + 44.598x_{18} \\
&+20.532x_{19} - 10.95x_{20} - 27 \geq \theta\,, \\
&37.512x_1 + 31.284x_2 + 27.828x_3 + 10.548x_4 + 30.864x_5 + 26.658x_6 \\
&+38.832x_7 + 11.04x_8 + 6.798x_9 + 10.032x_{10} + 12.738x_{11} + 5.688x_{12} \\
&+13.688x_{13} + 50.85x_{14} + 28.14x_{15} + 46.746x_{16} + 55.218x_{17} + 32.442x_{18} \\
&+28.704x_{19} + 2.82x_{20} - 25.2 \geq \theta\,, \\
&-33.928x_1 - 29.856x_2 - 41.632x_3 - 51.768x_4 - 29.984x_5 - 37.72x_6 \\
&-39.12x_7 - 34.56x_8 - 53.208x_9 - 33.024x_{10} - 33.328x_{11} - 39.304x_{12} \\
&-41.288x_{13} - 40.328x_{14} - 44.216x_{15} - 48.904x_{16} - 42.856x_{17} - 40.752x_{18} \\
&-48.496x_{19} - 43.344x_{20} + 40 \geq \theta\,, \\
&1.32x_1 + 6.18x_2 + 7.2x_3 + 3.9x_4 + 1.14x_5 + 3.36x_6 + 3x_7 + 3.72x_8 + 1.74x_9 \\
&+1.74x_{10} + 3.72x_{11} + 0.72x_{12} + 6.18x_{13} + 1.2x_{14} + 4.74x_{15} + 13.08x_{16} \\
&+1.68x_{17} + 6.18x_{18} + 11.52x_{19} + 4.86x_{20} - 6 \geq \theta\,, \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \\
&+x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,, \\
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} \\
&+y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,, \\
&x_i - 0.01y_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,, \\
&x_i - 0.55y_i \leq 0\,, \quad i = 1, 2, \ldots, 20\,, \\
&y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 20\,, \\
&x_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,.
\end{aligned}
$$

Table 6.9 Summary results of portfolio selection for conservative investor

Shape parameters & variables						Risk	Expected return		Liquidity
η	θ	α_{r36}	α_{r12}	α_w	α_L		Long term	Short term	
0.8948	2.1403	600	600	800	600	0.0473	0.0553	0.0646	0.0136
0.8848	2.0386	500	500	1000	500	0.0480	0.0571	0.0665	0.0141
0.8585	1.8028	400	400	1200	400	0.0485	0.0587	0.0682	0.0145

If the investor is not satisfied with the portfolio obtained, more portfolios can be generated by varying values of the shape parameters $\alpha_{r^{12}}$, $\alpha_{r^{36}}$, α_w and α_L in the problem P(6.4). Corresponding to three different sets of values of these parameters, the computational results are summarized in Table 6.9. Table 6.10 presents proportions of the assets in the obtained portfolios. The efficiency of the solutions is verified using the two-phase approach.

Table 6.10 The proportions of the assets in the obtained portfolios for conservative investor

Shape parameters				Allocation				
$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	ABB	ACC	AHB	ALL	BAL
600	600	800	600	0	0.4	0.02	0	0.019
500	500	1000	500	0	0.4	0.02	0	0.019
400	400	1200	400	0	0.4	0.02	0	0.019
				BEL	BHE	BPC	CIL	DRL
				0	0	0.028	0	0.026
				0	0	0.028	0	0.026
				0	0	0.028	0	0.026
				HIM	ITL	ITC	MTN	SIL
				0.1349	0	0	0	0
				0.1022	0	0	0	0
				0.074	0	0	0	0
				TPC	TIL	VOL	VSN	WIL
				0.3621	0	0	0.01	0
				0.3948	0	0	0.01	0
				0.422	0	0	0.01	0

A comparison of the solutions listed in Tables 6.5 and 6.9 highlights that if the investor chooses aggressive strategy then a higher level of expected returns is achieved than choosing conservative strategy, but it is accompanied with a higher risk level. On the other hand, if the investor prefers conservative strategy then a lower level of expected portfolio returns is achieved but it is accompanied with a lower level of portfolio risk.

Next, we show computational results by incorporating investor preferences. We consider the following three decision making situations.

- **Case 1**

Suppose the relative importance or weights of the fuzzy goals of expected short term return, expected long term return, risk and liquidity are $\omega_1 = 0.4$,

$\omega_2 = 0.35$, $\omega_3 = 0.15$, $\omega_4 = 0.1$, respectively. Setting $r_m^{12} = 0.05$, $r_m^{36} = 0.0475$, $w_m = 0.0525$ and $L_m = 0.0125$, we obtain a portfolio selection by solving problem P(6.5). The corresponding computational results are listed in Tables 6.11-6.12. The efficiency of the solution is verified by solving problem P(6.7) in the second-phase.

Table 6.11 Portfolio selection incorporating investor preferences (case 1)

$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	Risk	Expected return		Liquidity
					Long term	Short term	
600	600	800	600	0.0492	0.0606	0.0702	0.015

Table 6.12 The proportions of the assets in the obtained portfolio incorporating investor preferences (case 1)

	Allocation									
	ABB	ACC	AHB	ALL	BAL	BEL	BHE	BPC	CIL	DRL
Portfolio	0	0.4	0.02	0	0.019	0	0	0.028	0	0.026
	HIM	ITL	ITC	MTN	SIL	TPC	TIL	VOL	VSN	WIL
	0.04	0	0	0	0	0.457	0	0	0.01	0

The achievement levels of the various membership functions are $\eta_1 = 0.9999$, $\eta_2 = 0.9996$, $\eta_3 = 0.9334$, $\eta_4 = 0.8264$ which are consistent with the investor preferences. In other words, $(\eta_1 > \eta_2 > \eta_3 > \eta_4)$ agrees with $(\omega_1 > \omega_2 > \omega_3 > \omega_4)$.

• **Case 2**

Suppose the relative importance or weights of the fuzzy goals of expected short term return, expected long term return, risk and liquidity are $\omega_1 = 0.2$, $\omega_2 = 0.4$, $\omega_3 = 0.35$, $\omega_4 = 0.05$, respectively. Setting $r_m^{12} = 0.112$, $r_m^{36} = 0.075$, $w_m = 0.0525$ and $L_m = 0.0083$, we obtain a portfolio selection by solving problem P(6.5). The corresponding computational results are listed in Tables 6.13-6.14. The solution is found to be efficient using the two-phase approach.

The achievement levels of the various membership functions are $\eta_1 = 0.8081$, $\eta_2 = 0.9425$, $\eta_3 = 0.8916$, $\eta_4 = 0.126$ which are consistent with the investor preferences. In other words, $(\eta_2 > \eta_3 > \eta_1 > \eta_4)$ agrees with $(\omega_2 > \omega_3 > \omega_1 > \omega_4)$.

Table 6.13 Portfolio selection incorporating investor preferences (case 2)

$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	Risk	Expected return		Liquidity
					Long term	Short term	
600	600	800	600	0.0499	0.0797	0.114	0.0051

Table 6.14 The proportions of the assets in the obtained portfolio incorporating investor preferences (case 2)

	Allocation									
	ABB	ACC	AHB	ALL	BAL	BEL	BHE	BPC	CIL	DRL
Portfolio	0	0.2	0	0	0.019	0.025	0.02	0	0	0
	HIM	ITL	ITC	MTN	SIL	TPC	TIL	VOL	VSN	WIL
	0	0	0	0	0	0.016	0.678	0.032	0.01	0

Table 6.15 Portfolio selection incorporating investor preferences (case 3)

$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	Risk	Expected return		Liquidity
					Long term	Short term	
600	600	800	600	0.056	0.0811	0.1051	0.0109

Table 6.16 The proportions of the assets in the obtained portfolio incorporating investor preferences (case 3)

	Allocation									
	ABB	ACC	AHB	ALL	BAL	BEL	BHE	BPC	CIL	DRL
Portfolio	0	0	0.02	0	0.019	0	0.02	0	0	0
	HIM	ITL	ITC	MTN	SIL	TPC	TIL	VOL	VSN	WIL
	0	0	0.023	0	0	0.385	0.491	0.032	0.01	0

- **Case 3**

Suppose the relative importance or weights of the fuzzy goals of expected short term return, expected long term return, risk and liquidity are $\omega_1 = 0.2$, $\omega_2 = 0.3$, $\omega_3 = 0.35$, $\omega_4 = 0.15$, respectively. Setting $r_m^{12} = 0.1$, $r_m^{36} = 0.075$, $w_m = 0.0625$ and $L_m = 0.0125$, we obtain a portfolio selection by solving problem P(6.5). The corresponding computational results are listed in Tables 6.15-6.16. The solution is found to be efficient using the two-phase approach.

The achievement levels of the various membership functions are $\eta_1 = 0.9526$, $\eta_2 = 0.9749$, $\eta_3 = 0.9945$, $\eta_4 = 0.2769$ which are consistent with the investor preferences. In other words, $(\eta_3 > \eta_2 > \eta_1 > \eta_4)$ agrees with $(\omega_3 > \omega_2 > \omega_1 > \omega_4)$.

6.4 Comments

In this chapter, we have presented the following facts:

- Multi-criteria portfolio selection using fuzzy mathematical programming has been discussed.
- The fuzzy methodology has been used to incorporate uncertainty into historical data and also to incorporate subjective/intutive characteristics into the portfolio selection models, which are basic aspects for establishing different estimations of investor preferences.
- The generalization of the semi-absolute deviation portfolio optimization model using nonlinear S-shape membership functions for the investor's aspiration levels has been discussed.
- The computational results based on real-world data have been provided to demonstrate the effective working of the portfolio selection models. Further, the efficiency of the obtained solutions has been verified using a two-phase approach.
- The advantage of the portfolio selection models have been shown under the situation that if the investor is not satisfied with any of the portfolio obtained, more portfolios can be generated by varying values of shape parameters of the nonlinear S-shape membership functions.
- Moreover, it has been shown that the fuzzy portfolio selection models can provide satisfying portfolio selection strategies according to the investor's vague aspiration levels, varying degree of satisfaction and varying importance of various objectives.

Chapter 7
Suitability Considerations in Multi-criteria Fuzzy Portfolio Optimization-I

Abstract. In this chapter, we present fuzzy framework of portfolio selection by simultaneous consideration of suitability and optimality. Suitability is a behavioral concept that refers to the propriety of the match between investor preferences and portfolio characteristics. The approach described in this chapter for portfolio selection is based on multiple methodologies. We evolve a typology of investors using the inputs from a primary survey of investor preferences. A cluster analysis is done on the basis of three evaluation indices to categorize the chosen sample of financial assets into different clusters. Further, using analytical hierarchy process (AHP), we determine weights of the various assets within a cluster from the point of view of the investor preferences. The optimal asset allocation based on a mix of suitability and optimality is obtained using fuzzy portfolio selection models. The criteria used for portfolio selection are short term return, long term return, risk, liquidity and AHP weighted score of suitability.

7.1 Overview of AHP

AHP [104] is a decision making approach widely used for multi-criteria decision making problems in a number of application domains. It involves simple procedure and is accessible to the end user as well as it possesses meaningful scientific justifications. The important advantages of AHP are its simplicity, robustness and the ability to incorporate 'intangibles' into the decision making process. The decision maker judges the importance of each criterion using pair-wise comparisons. The outcome of AHP is the weight of each decision alternative. The following three steps are followed for solving decision making problems by AHP.

Step 1: Establishing Structural Hierarchy
This step allows a complex decision to be structured into a hierarchy descending from an overall objective to various 'criteria', 'subcriteria' and so on until the lowest level. The objective or the overall goal of the decision is

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represented at the top level of the hierarchy. The criteria and subcriteria contributing to the decision are represented at the intermediate levels. Finally, the decision alternatives or selection choices are laid down at the bottom level of the hierarchy. According to Saaty [104], a hierarchy can be constructed by creative thinking, recollection and using people's perspectives. The structure of a hierarchy depends upon the nature or type of managerial decisions. Also, the number of the levels in a hierarchy depends on the complexity of the problem being analyzed and the degree of detail of the problem that an analyst requires. As such, the hierarchy representation of a decision process may vary from one person to another.

Step 2: Establishing Comparative Judgements
Once the hierarchy has been structured, the next step is to determine the priorities of elements at each level ('element' here means every member of the hierarchy). A set of pair-wise comparison matrices of all elements in a level of the hierarchy with respect to an element of the immediate higher level are constructed so as to prioritize and convert individual comparative judgements into ratio scale measurements. The preferences are quantified by using a nine-point scale [104], see Table 7.1. The pair-wise comparisons are given in terms of how much more an element A is important than an element B.

Table 7.1 Saaty's scale for pair-wise comparisons

Verbal scale	Numerical values
Equally important, likely or preferred	1
Moderately more important, likely or preferred	3
Strongly more important, likely or preferred	5
Very strongly more important, likely or preferred	7
Extremely more important, likely or preferred	9
Intermediate values to reflect compromise	2,4,6,8
Reciprocals for inverse comparison	Reciprocals

Step 3: Synthesis of Priorities and the Measurement of Consistency
The elements of each level of the decision hierarchy are rated using pair-wise comparison. After all the elements have been compared pair by pair, a paired comparison matrix is formed. The order of the matrix depends on the number of elements at each level. The number of such matrices at each level depends on the number of elements at the immediate upper level that it links to. After developing all the paired comparison matrices, the eigenvector or the relative weights representing the degree of the relative importance amongst the elements and the maximum eigenvalue (λ_{max}) are calculated for each matrix. The λ_{max} value is an important validating parameter which is used

as a reference index to screen information by calculating the consistency ratio of the estimated vector in order to validate whether the paired comparison matrix provides a completely consistent evaluation. The consistency ratio is calculated as per the following steps

(a) Calculate the eigenvector or the relative weights and λ_{max} for each matrix of order n.
(b) Compute the consistency index (CI) for each matrix of order n as follows:
$$CI = (\lambda_{max} - n)/(n - 1).$$
(c) The consistency ratio (CR) is calculated as follows:
$$CR = CI/RI,$$

where RI is a known random consistency index that has been obtained from a large number of simulation runs and varies according to the order of matrix. If CI is sufficiently small then pair-wise comparisons are probably consistent enough to give useful estimates of the weights. The acceptable CR value for a matrix at each level is less than or equal to 0.1, i.e., if $CI/RI \leq 0.10$ then the degree of consistency is satisfactory; however, if $CI/RI > 0.10$ then serious inconsistencies may exist and hence AHP may not yield meaningful results. The evaluation process should, therefore, be reviewed and improved. The eigenvectors are used to calculate the global weights if there is an acceptable degree of consistency of the selection criteria.

7.2 Suitability Considerations

Financial experts and investment companies use various techniques to profile investors and then recommend a suitable asset allocation. In our view, portfolio selection models can be substantially improved by incorporating investor preferences. Some of the relevant references on portfolio selection by simultaneous consideration of suitability and optimality include [11, 42, 43].

7.2.1 Investor Typology

No two investors are alike. Investor diversity is characterized by variations in the demographic, socio-cultural, economic and psychographic factors, each factor comprising in turn, a host of interrelated variables. Note that there are many variables that impinge upon investment decision making and any list would at best be illustrative. Moreover, it is not just an individual variable but a configuration of several variables that influence investor behavior. Since these variables are not static, their relative influence changes with the passage of time. Therefore, it is better to capture the dynamic nature of these variables through life-stage analysis where the time element is broken into discrete stages of life to reflect the combined effect of several variables. Investment experts map the investors on these variables to assess investment preferences. Such an assessment facilitates recommendation of the appropriate investment alternatives that have, at least, a *prima facie* suitability for the investors. The individual investor then chooses from among these alternatives.

We capture the investor diversity in terms of a typology developed from a primary survey [40]. The survey relied on structured questionnaires covering a variety of interrelated aspects such as investors' economic and financial position-including income and types of investment held, past experiences and future investment intentions. Analysis of the survey data showed variation in investor behavior across these variables. From a behavioral perspective such a variation is understandable as even though investors prefer more of return and liquidity over less and less of risk over more, they would order these investment objectives differently. The profiling of the investors is done on the basis of a select variables that are discussed as under.

- **Age**

It is generally believed that younger investors take greater risks in anticipation of higher returns. With growing age, they rebalance portfolio in favor of safer and more secured even though somewhat lower returns. The investor population can be broadly categorized as follows:

Young (25-40 years); Middle-aged (40-55 years); Seniors (above 55 years)

- **Gender**

On the basis of their gender, investors may be categorized into:

Male; Female

In finance literature, it is believed that women invest more conservatively and are less likely to hold risky assets than men. They would not rebalance their portfolios frequently and would prefer a buy-and-hold strategy.

- **Life-stage**

Life-stage analysis is a well-entrenched strategy for understanding consumption, savings and investment behavior. A person's economic behavior is likely to change along with changes in the stages of life, i.e., whether a person is

Single; Married without dependents (children/adults);

Married/single with dependents; *Retired persons*

The life-stage analysis leads to a more meaningful interpretation of investor behavior. Investors' income level, saving potential, time horizon of investment and risk appetite, all depend on the stages of the life-cycles.

- **Income status**

Both consumption and savings are related to income; as income increases, keeping other things constant, savings are likely to grow at a faster rate than consumption. Thus, it is likely that persons from middle income and upper-middle income groups (however defined) are drawn to investment markets. As for high net-worth, affluent investors, it may more be a matter of wealth preservation rather than wealth creation. The survey that we have relied on for evolving the investor typology reports, that, by and large the sample comprised middle and upper-middle class households with the corresponding monthly incomes as follows:

Low/lower-middle income group (up to 25,000.00 INR);
Middle income group (25,001.00 INR - 40, 000.00 INR);
Upper-middle/upper income group (above 40,000.00 INR)
where INR stands for Indian National Rupee.

- **Accumulated savings/wealth**
Size of the accumulated savings or the wealth status has a tremendous bearing on the investor behavior. A wealthy investor can afford to undertake higher risk. However, the relationship between wealth and return/risk expectations is not straightforward. It is also possible that wealthy investors seek to protect and conserve wealth rather than risk it any further.

- **Education**
Awareness about the investment alternatives as well as ability to arrive at an informed investment decision is a matter of education. Professionally qualified persons are more likely to take better-informed decisions and discriminate among various investment alternatives. They take calculated risks whereas their lesser-educated counterparts show greater aversion towards risk. The relevant classes that investors were categorized into as per the survey data are: non-graduates and graduates & above. However, we are of the view that professionals may be categorized separately as they comprise a class unto themselves. The resultant categories hence would be

Non-graduates; Graduates & above; Professionally qualified

- **Occupation**
A person's occupational profile has a bearing on his/her investment behavior. Employment in corporate and multinational sectors provides exposure to share-owning culture. In fact, many companies now-a-days offer stock options and the employees of such companies, therefore, get into the habit of investing in stock markets. Persons in government service are mentally trained to be risk averse and look for secured investment options. Business persons resort to non-business investment alternatives as means of investing their funds in short-term investments and look for quicker and more liquid returns. Self-employed professionals prefer to take calculated risks and seek to accumulate wealth through standard savings and investment plans. Accordingly, and in line with the survey, the investors may be categorized as per their occupational profile into:

Service: large private and multinational companies; *Service: government*;
Business persons; Self-employed professionals

- **Prior experience**
A prior experience in the investment markets has a moderating impact on investor expectations. Experienced investors have more reasonable expectations from their investments. The relatively new entrants have great expectations

and are more likely to get carried away by the swings of the market. Most of the investors covered in the survey had some prior experience in investment. Many of them had a prior experience of 10 years or more. We categorize the investor experience as follows:

Less than a year; 1-5 years; *6-10 years; 10 years and above*

Since investment decision is the outcome of a configuration of several variables, as an investment expert what one attempts is a broad estimation of the investor type. We identify three stereotypes from an overview of the aforesaid variables:

Return seekers: young, males, single, professionally qualified, corporate executives;

Safety seekers: females, middle-aged, mid-career, having dependents, government servants, graduates or lesser educated, self-employed persons;

Liquidity seekers: seniors, retired persons, business persons.

Categorization of investors as above is useful for incorporating suitability considerations into portfolio selection despite the fact that such a stereotyping may not be all inclusive. The benefit of stereotyping, however, is that it creates a *prima facie* ground for recommending the investment alternatives.

7.2.2 Modeling Suitability with the AHP

We measure suitability of the assets as per investor preferences using AHP. The AHP model used here is a four level hierarchy as shown in Fig. 7.1. Level 1 represents the goal, i.e., Suitability of assets; level 2 represents the three main criteria of Suitability: Income and Savings (IS), Investment Objectives (IO) and Investing Experience (IE). At level 3, these criteria are decomposed into various subcriteria, i.e., IS is decomposed into Income (IN), Source (SO), Savings (SA) and Saving Rate (SR); IO is decomposed into Age (AG), Dependents (DE), Time Horizon (TH) and Risk/Loss Appetite (R/L); IE is decomposed into Length of Prior Experience (LE), Equity Holding (EH) and Education (ED). Finally, the bottom level of the hierarchy, i.e., level 4, represents the alternatives (assets). The said criteria and subcriteria of suitability have been selected on the basis of the primary survey of investor preferences [40].

7.3 Portfolio Selection Based on Suitability and Optimality

Assume that investors allocate their wealth among n assets offering random rates of return. We introduce some notation as follows:

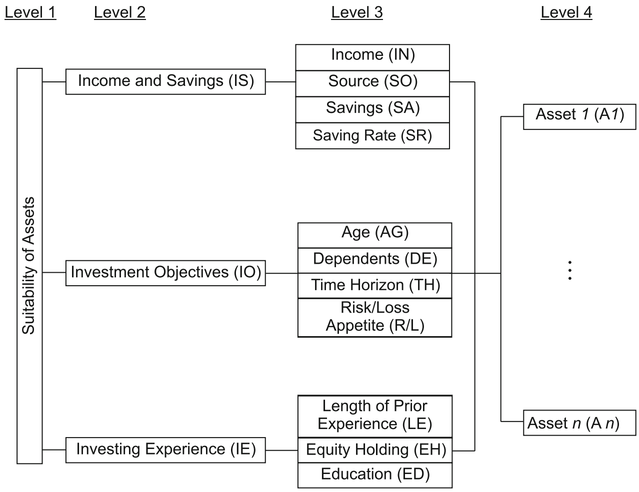

Fig. 7.1 Structural hierarchy for suitability of assets

7.3.1 Notation

r_i: the expected rate of return of the i-th asset,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

r_i^{12}: the average performance of the i-th asset during a 12-month period,

r_i^{36}: the average performance of the i-th asset during a 36-month period,

r_{it}: the historical return of the i-th asset over the past period t,

w_{AHP_i}: the AHP suitability weight of the i-th asset,

u_i: the maximal fraction of the capital allocated to the i-th asset,

l_i: the minimal fraction of the capital allocated to the i-th asset,

L: the minimum desired level of portfolio liquidity,

$\tilde{L}_i$: the fuzzy turnover rate of the i-th asset,

h: the number of assets held in the portfolio,

T: the total time span.

We consider the following objective functions and constraints in the multiobjective portfolio selection problem.

7.3.2 Objective Functions

Short Term Return

The short term return of the portfolio is expressed as

$$f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i ,$$

where $r_i^{12} = \frac{1}{12}\sum_{t=1}^{12} r_{it}$, $i = 1, 2, \ldots, n$.

Long Term Return

The long term return of the portfolio is expressed as

$$f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i ,$$

where $r_i^{36} = \frac{1}{36}\sum_{t=1}^{36} r_{it}$, $i = 1, 2, \ldots, n$.

AHP Weighted Score of Suitability

The AHP weighted score of suitability of the portfolio is expressed as

$$f_3(x) = \sum_{i=1}^{n} w_{AHP_i} x_i .$$

Risk

The portfolio risk using semi-absolute deviation measure is expressed as

$$f_4(x) = w(x) = \frac{1}{T}\sum_{t=1}^{T} w_t(x) = \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2T} .$$

Liquidity
The portfolio liquidity is expressed as

$$f_5(x) = E(\tilde{L}(x)) = E\left(\sum_{i=1}^{n} \tilde{L}_i x_i\right) = \sum_{i=1}^{n} \left(\frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6}\right) x_i \,.$$

7.3.3 Constraints

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1 \,.$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \le u_i y_i \,, \quad i = 1, 2, \ldots, n \,.$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \ge l_i y_i \,, \quad i = 1, 2, \ldots, n \,.$$

Number of assets held in the portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h \,.$$

No short selling of assets is expressed as

$$x_i \ge 0 \,, \quad i = 1, 2, \ldots, n \,.$$

7.3.4 The Decision Problem

The multiobjective mixed integer nonlinear programming problem for portfolio selection based on suitability and optimality is formulated as follows:

$$\mathbf{P(7.1)}\quad \max\ f_1(x)=\sum_{i=1}^{n} r_i^{12}x_i$$

$$\max\ f_2(x)=\sum_{i=1}^{n} r_i^{36}x_i$$

$$\max\ f_3(x)=\sum_{i=1}^{n} w_{AHP_i}x_i$$

$$\min\ f_4(x)=w(x)=\sum_{t=1}^{T}\frac{\left|\sum_{i=1}^{n}(r_{it}-r_i)x_i\right|+\sum_{i=1}^{n}(r_i-r_{it})x_i}{2T}$$

$$\max\ f_5(x)=E(\tilde{L}(x))=\sum_{i=1}^{n}\left(\frac{L_{a_i}+L_{b_i}}{2}+\frac{L_{\beta_i}-L_{\alpha_i}}{6}\right)x_i$$

subject to

$$\sum_{i=1}^{n} x_i = 1\,,$$

$$\sum_{i=1}^{n} y_i = h\,,$$

$$x_i \le u_i y_i\,,\quad i=1,2,\ldots,n\,,$$

$$x_i \ge l_i y_i\,,\quad i=1,2,\ldots,n\,,$$

$$x_i \ge 0\,,\quad i=1,2,\ldots,n\,,$$

$$y_i \in \{0,1\}\,,\quad i=1,2,\ldots,n\,.$$

To eliminate the absolute-valued function in problem P(7.1), we transform the problem into the following multiobjective mixed integer linear programming problem

$$\mathbf{P(7.2)}\quad \max\ f_1(x)=\sum_{i=1}^{n} r_i^{12}x_i$$

$$\max\ f_2(x)=\sum_{i=1}^{n} r_i^{36}x_i$$

$$\max\ f_3(x)=\sum_{i=1}^{n} w_{AHP_i}x_i$$

$$\min\ f_4(p)=w(p)=\frac{1}{T}\sum_{t=1}^{T} p_t$$

$$\max\ f_5(x) = E(\tilde{L}(x)) = \sum_{i=1}^{n}\left(\frac{L_{a_i}+L_{b_i}}{2} + \frac{L_{\beta_i}-L_{\alpha_i}}{6}\right)x_i$$

subject to

$$p_t + \sum_{i=1}^{n}(r_{it}-r_i)x_i \geq 0, \quad t = 1,2,\ldots,T,$$
$$\sum_{i=1}^{n} x_i = 1\,,$$
$$\sum_{i=1}^{n} y_i = h\,,$$
$$x_i \leq u_i y_i\,, \quad i = 1,2,\ldots,n\,,$$
$$x_i \geq l_i y_i\,, \quad i = 1,2,\ldots,n\,,$$
$$x_i \geq 0\,, \quad i = 1,2,\ldots,n\,,$$
$$p_t \geq 0\,, \quad t = 1,2,\ldots,T\,,$$
$$y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,n\,.$$

7.4 Fuzzy Portfolio Selection Models Based on Suitability and Optimality

We use logistic function [119], i.e., a nonlinear S-shape membership function to express vague aspiration levels of the investor. The following nonlinear S-shape membership function is used to characterize the goal of expected short term return

- $$\mu_{r^{12}}(x) = \frac{1}{1+\exp\left(-\alpha_{r^{12}}\left(\sum_{i=1}^{n} r_i^{12}x_i - r_m^{12}\right)\right)},$$

where r_m^{12} is the mid-point (middle aspiration level for the expected short term return) at which the membership function value is 0.5 and $\alpha_{r^{12}}$ is provided by the investor based on his/her degree of satisfaction of the goal.

Similarly, we define membership functions of the goals of expected long term return ($\mu_{r^{36}}(x)$), AHP weighted score of suitability ($\mu_{w_{AHP}}(x)$) and liquidity ($\mu_{\tilde{L}}(x)$) as follows:

- $$\mu_{r^{36}}(x) = \frac{1}{1+\exp\left(-\alpha_{r^{36}}\left(\sum_{i=1}^{n} r_i^{36}x_i - r_m^{36}\right)\right)},$$
- $$\mu_{w_{AHP}}(x) = \frac{1}{1+\exp\left(-\alpha_{w_{AHP}}\left(\sum_{i=1}^{n} w_{AHP_i}x_i - w_{AHP_m}\right)\right)},$$

- $$\mu_{\tilde{L}}(x) = \frac{1}{1 + \exp\left(-\alpha_L(E(\tilde{L}(x)) - L_m)\right)},$$

where r_m^{36}, w_{AHP_m}, L_m are the respective mid-points and $\alpha_{r^{36}}$, $\alpha_{w_{AHP}}$, α_L are provided by the investor.

The membership function of the goal of risk is given by

- $$\mu_w(x) = \frac{1}{1 + \exp\left(\alpha_w(w(x) - w_m)\right)},$$

where w_m is the mid-point and α_w is provided by the investor based on his/her degree of satisfaction regarding the level of risk.

Using Bellman-Zadeh's maximization principle [7], the fuzzy portfolio selection problem is formulated as follows:

$$
\begin{aligned}
\textbf{P(7.3)} \quad & \max \ \eta \\
& \text{subject to} \\
& \eta \le \mu_{r^{12}}(x), \\
& \eta \le \mu_{r^{36}}(x), \\
& \eta \le \mu_{w_{AHP}}(x), \\
& \eta \le \mu_w(x), \\
& \eta \le \mu_{\tilde{L}}(x), \\
& \sum_{i=1}^{n} x_i = 1, && (7.1) \\
& \sum_{i=1}^{n} y_i = h, && (7.2) \\
& x_i \le u_i y_i, \quad i = 1, 2, \ldots, n, && (7.3) \\
& x_i \ge l_i y_i, \quad i = 1, 2, \ldots, n, && (7.4) \\
& x_i \ge 0, \quad i = 1, 2, \ldots, n, && (7.5) \\
& y_i \in \{0, 1\}, \quad i = 1, 2, \ldots, n, && (7.6) \\
& 0 \le \eta \le 1. && (7.7)
\end{aligned}
$$

The problem P(7.3) is a mixed integer nonlinear programming problem. The above problem can be transformed into the following mixed integer linear programming problem using the transformation $\theta = \log \dfrac{\eta}{1-\eta}$.

$$
\begin{aligned}
\mathbf{P(7.4)}\quad & \max\ \theta \\
& \text{subject to} \\
& \theta \le \alpha_{r^{12}}\left(\sum_{i=1}^{n} r_i^{12}x_i - r_m^{12}\right), \\
& \theta \le \alpha_{r^{36}}\left(\sum_{i=1}^{n} r_i^{36}x_i - r_m^{36}\right), \\
& \theta \le \alpha_{w_{AHP}}\left(\sum_{i=1}^{n} w_{AHP_i}x_i - w_{AHP_m}\right), \\
& \theta \le \alpha_{w}\left(w_m - w(x)\right), \\
& \theta \le \alpha_{L}\left(\sum_{i=1}^{n}\left(\frac{L_{a_i}+L_{b_i}}{2}+\frac{L_{\beta_i}-L_{\alpha_i}}{6}\right)x_i - L_m\right), \\
& \text{and Constraints (7.1)} - \text{(7.6)}.
\end{aligned}
$$

Note that $\theta \in\ ]-\infty,+\infty[$. The absolute-valued function in the expression of $w(x)$ can be eliminated on the same lines as discussed in Section 7.3.4.

To incorporate relative importance of various fuzzy objectives in portfolio selection, the weighted additive model of the fuzzy portfolio selection problem is written as follows:

$$
\begin{aligned}
\mathbf{P(7.5)}\quad & \max\ \sum_{p=1}^{5}\omega_p\eta_p \\
& \text{subject to} \\
& \eta_1 \le \mu_{r^{12}}(x), \\
& \eta_2 \le \mu_{r^{36}}(x), \\
& \eta_3 \le \mu_{w_{AHP}}(x), \\
& \eta_4 \le \mu_{w}(x), \\
& \eta_5 \le \mu_{\tilde{L}}(x), \\
& 0 \le \eta_p \le 1, \qquad p = 1,2,\ldots,5, \\
& \text{and Constraints (7.1)} - \text{(7.6)},
\end{aligned}
$$

where ω_p is the relative weight of the p-th objective given by investors such that $\omega_p > 0$ and $\sum_{p=1}^{5}\omega_p = 1$.

Further, in order to ensure efficiency of the obtained solution, we solve the problems P(7.6) and P(7.7) corresponding to the problems P(7.4) and P(7.5) respectively, in the second-phase.

$$
\begin{aligned}
\mathbf{P(7.6)}\ \ & \max \sum_{p=1}^{5} \omega_p \theta_p \\
& \text{subject to} \\
& \log \frac{\mu_{r^{12}}(x^*)}{1-\mu_{r^{12}}(x^*)} \le \theta_1 \le \alpha_{r^{12}} \left(\sum_{i=1}^{n} r_i^{12} x_i - r_m^{12} \right), \\
& \log \frac{\mu_{r^{36}}(x^*)}{1-\mu_{r^{36}}(x^*)} \le \theta_2 \le \alpha_{r^{36}} \left(\sum_{i=1}^{n} r_i^{36} x_i - r_m^{36} \right), \\
& \log \frac{\mu_{w_{AHP}}(x^*)}{1-\mu_{w_{AHP}}(x^*)} \le \theta_3 \le \alpha_{w_{AHP}} \left(\sum_{i=1}^{n} w_{AHP_i} x_i - w_{AHP_m} \right), \\
& \log \frac{\mu_{w}(x^*)}{1-\mu_{w}(x^*)} \le \theta_4 \le \alpha_{w} \left(w_m - w(x) \right), \\
& \log \frac{\mu_{\tilde{L}}(x^*)}{1-\mu_{\tilde{L}}(x^*)} \le \theta_5 \le \alpha_{L} \left(\sum_{i=1}^{n} \left(\frac{L_{a_i} + L_{b_i}}{2} + \frac{L_{\beta_i} - L_{\alpha_i}}{6} \right) x_i - L_m \right), \\
& \text{and Constraints } (7.1) - (7.6),
\end{aligned}
$$

where x^* is an optimal solution of problem P(7.4), $\omega_1 = \ldots = \omega_5$, $\omega_p > 0$, $\sum_{p=1}^{5} \omega_p = 1$ and $\theta_p \in \,]-\infty, +\infty[\,,\ p = 1, 2, \ldots, 5.$.

$$
\begin{aligned}
\mathbf{P(7.7)}\ \ & \max \sum_{p=1}^{5} \omega_p \eta_p \\
& \text{subject to} \\
& \mu_{r^{12}}(x^{**}) \le \eta_1 \le \mu_{r^{12}}(x), \\
& \mu_{r^{36}}(x^{**}) \le \eta_2 \le \mu_{r^{36}}(x), \\
& \mu_{w_{AHP}}(x^{**}) \le \eta_3 \le \mu_{w_{AHP}}(x), \\
& \mu_{w}(x^{**}) \le \eta_4 \le \mu_{w}(x), \\
& \mu_{\tilde{L}}(x^{**}) \le \eta_5 \le \mu_{\tilde{L}}(x), \\
& 0 \le \eta_p \le 1, \quad p = 1, 2, \ldots, 5, \\
& \text{and Constraints } (7.1) - (7.6),
\end{aligned}
$$

where x^{**} is an optimal solution of problem P(7.5), ω_p is the relative weight of the p-th objective given by the investor such that $\omega_p > 0$ and $\sum_{p=1}^{5} \omega_p = 1$.

7.5 Numerical Illustration

Here, we present the results of an empirical study based on the data set of daily closing prices of 150 assets listed on NSE, Mumbai, India.

7.5.1 Asset Clusters

Just as different investor types show distinct ordering of return, risk and liquidity criteria, likewise different assets also show distinct characteristics vis-à-vis these criteria. Thus, it is desirable to stratify the assets into clusters on the basis of some pre-defined characteristics. We consider three evaluation indices, namely, average return, standard deviation denoting risk and turnover rate denoting liquidity. Since the measurement units and scales of all the three indices are not same, we perform normalization using z-score transformation. We use K-means method [68] for clustering of the assets. In order to find the most suitable number of clusters (k) for the input data set, we rely on the silhouette coefficients [68]. The silhouette coefficient $s(i)$ is computed as per the following steps

(a) For the i-th object, calculate its average distance to all other objects in its cluster; call this value a_i.
(b) For the i-th object and any cluster not containing the object, calculate the object's average distance to all the objects in the given cluster. Find the minimum such value with respect to all the clusters; call this value b_i.
(c) For the i-th object, the silhouette coefficient is $s(i) = \dfrac{b_i - a_i}{\max(a_i, b_i)}$.

The value of the silhouette coefficient of an object can vary between -1 and 1, which indicates how much that object belongs to the cluster in which it is classified. The closer the value is to 1, the higher the degree that the object belongs to its cluster. The silhouette coefficients are used here to quantify the quality of assignment of an asset to a particular cluster. The silhouette value of a cluster is the average of the silhouette coefficients of all data items belonging to the cluster. We refer to the following interpretation of the silhouette value of a cluster proposed by Kaufman and Rousseeuw [68].

$0.71 \leq$ cluster silhouette ≤ 1 means it is a strong cluster;

$0.51 \leq$ cluster silhouette ≤ 0.7 means it is a reasonable cluster;

$0.26 \leq$ cluster silhouette ≤ 0.5 means it is a weak or artificial cluster;

cluster silhouette < 0.25 means no cluster is found.

The silhouette value for k is the average silhouette values of the k clusters. The most suitable k is the one with the highest average silhouette value.

Using K-means tool of the MATLAB 7.0 software, we experiment with a range of values for k to perform cluster analysis. We find that the average silhouette value 0.6841 for $k = 3$ is the highest in comparison with other values of k (see Figs. 7.2-7.5). To overcome local minima, we use the optional 'replicates' parameter. The computational results are summarized in Table 7.2 in which the mean value of each variable (index) is provided.

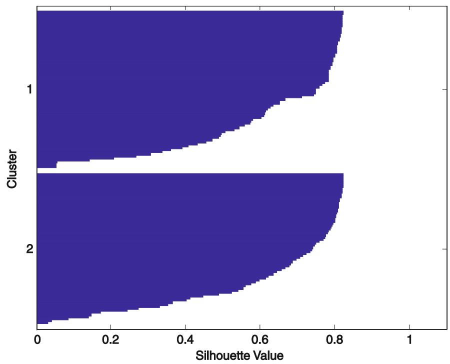

Fig. 7.2 Cluster analysis for $k = 2$ (average silhouette = 0.5129)

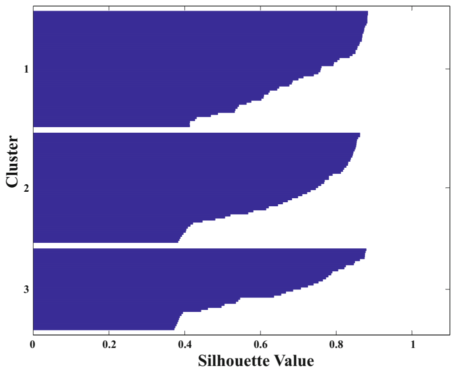

Fig. 7.3 Cluster analysis for $k = 3$ (average silhouette = 0.6841)

Table 7.2 Results of cluster analysis

Variables	Clusters		
	Cluster 1 (56 assets)	Cluster 2 (51 assets)	Cluster 3 (43 assets)
Average return	0.14084	0.27078	0.18015
Standard deviation	0.44474	0.52338	0.32999
Turnover rate	0.011	0.00176	0.00487
Category	Liquid assets	High-yield assets	Less risky assets

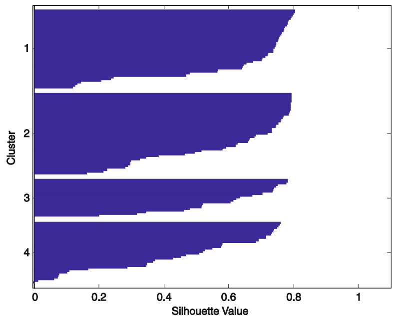

Fig. 7.4 Cluster analysis for $k = 4$ (average silhouette = 0.4721)

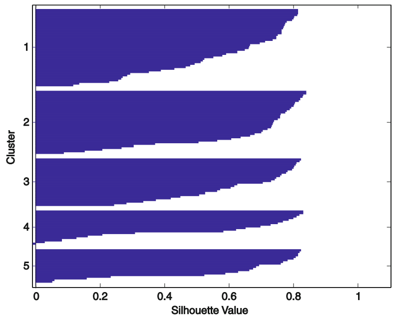

Fig. 7.5 Cluster analysis for $k = 5$ (average silhouette = 0.2826)

On the basis of computational results, we propose the following three clusters of assets:

(i) ***Cluster 1: liquid assets***
Assets in cluster 1 are categorized as liquid assets since mean value for liquidity is the highest in this cluster. This cluster is typified by low but widely varying returns.

(ii) ***Cluster 2: high-yield assets***
Assets in cluster 2 are categorized as high-yielding ones since they have rather high returns. On the expected lines of return/risk relationship, these assets also show high standard deviation. Although, investors may gain from the high returns they also have to endure the high risk. These assets have low liquidity amongst all the clusters indicating that high-yielding investment involves a longer time horizon.

(iii) ***Cluster 3: less risky assets***
Assets in cluster 3 are categorized as less risky assets since compared to the other clusters these assets have the lowest standard deviation. The return is not high but medium. The liquidity is medium too.

The asset clusters discussed above have a *prima facie* suitability for the corresponding investor types identified in Section 7.2.1. Thus, the cluster of liquid assets is suitable for liquidity seekers, the cluster of high-yield assets is suitable for return seekers and the cluster of less risky assets is suitable for safety seekers.

7.5.2 Calculation of AHP Weights

The next step is to measure suitability of the assets from a particular cluster as per investor preferences. Here, we follow the AHP model described in Section 7.2.2. For the sake of simplicity, we make the following assumptions.

(a) Rather than finding suitability of all the assets in a given cluster, the procedure is implemented for randomly chosen set of 20 assets from each cluster.
(b) Rather than accounting for investor diversity at each level of the AHP hierarchy, the same is introduced at the bottom level of the hierarchy.

Note that the above assumptions are all relaxable and the AHP model is adequately capable of handling the same. The procedure followed is a pair-wise comparison of the criteria, subcriteria and the assets, refer to Fig. 7.1 for complete hierarchy. At level 2, we determine local weights of the criteria with respect to the overall goal of suitability of assets at level 1. At level 3, we determine local weights of the subcriteria with respect to their respective parent criterion at level 2. For example, the subcriteria-income, source, savings and saving rate are pair-wise compared with respect to their parent criterion-income and savings. For the data in respect of pair-wise comparison matrices at levels 2 and 3, we have relied on inputs from investment experts on the Saaty's verbal scale (see Table 7.1). As noted earlier, we have not introduced investor diversity as yet. Hence, at these two levels the local weights of the criteria and the subcriteria are identical for all the three investor types. At level 4, we determine the local weights of each of the 20 assets with respect to each of the eleven subcriteria at level 3. At this stage, we account for investor diversity and incorporate the same by taking the investor preferences on the Saaty's verbal

Table 7.3 AHP weights of 20 assets from cluster 1

Criteria	Subcriteria	Weights	Global weights									
			A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
IS												
0.571429	IN	0.277630	0.014398	0.012817	0.013041	0.004833	0.002722	0.004849	0.005537	0.009376	0.003553	0.013162
	SO	0.063455	0.003106	0.002921	0.001481	0.001599	0.000727	0.001006	0.001067	0.001129	0.000795	0.003106
	SA	0.555260	0.012630	0.025975	0.020786	0.013779	0.003868	0.013432	0.006585	0.021974	0.004601	0.006774
	SR	0.103655	0.002681	0.001504	0.003659	0.003859	0.008799	0.001425	0.002604	0.001396	0.006890	0.005126
IO												
0.285714	AG	0.249563	0.002796	0.006612	0.002964	0.002754	0.001133	0.003018	0.002795	0.001654	0.001137	0.002765
	DE	0.547768	0.013465	0.022696	0.021579	0.005481	0.003135	0.004544	0.004965	0.005125	0.003573	0.006831
	TH	0.127614	0.001681	0.000720	0.000740	0.000905	0.004194	0.001578	0.003293	0.000838	0.004333	0.001731
	R/L	0.075056	0.000846	0.001621	0.000860	0.000830	0.000238	0.000851	0.000442	0.001439	0.000340	0.000375
IE												
0.142857	LE	0.557143	0.003969	0.002058	0.002082	0.003559	0.001175	0.003969	0.003934	0.003552	0.002211	0.003911
	EH	0.122619	0.000805	0.000467	0.001421	0.001487	0.002644	0.000431	0.000777	0.000459	0.002125	0.001428
	ED	0.320238	0.002747	0.001414	0.001414	0.004989	0.000985	0.002399	0.004788	0.002714	0.004952	0.002581
		AHP weights	0.059124	0.078805	0.070026	0.044074	0.029619	0.037502	0.036787	0.049657	0.034510	0.047791

Criteria	Subcriteria	Weights	Global weights									
			A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
IS												
0.571429	IN	0.277630	0.007258	0.004888	0.004495	0.006771	0.004533	0.013796	0.007447	0.013796	0.007632	0.003740
	SO	0.063455	0.001524	0.000967	0.001105	0.001802	0.001342	0.002960	0.001667	0.005236	0.001572	0.001150
	SA	0.555260	0.012346	0.031729	0.013116	0.020356	0.030766	0.018903	0.030731	0.003980	0.005108	0.019855
	SR	0.103655	0.002968	0.000925	0.001553	0.003035	0.002371	0.002614	0.001510	0.002497	0.000901	0.002915
IO												
0.285714	AG	0.249563	0.003033	0.007466	0.002893	0.005184	0.007064	0.005451	0.006534	0.000748	0.001053	0.004251
	DE	0.547768	0.006867	0.004393	0.004654	0.007407	0.004820	0.007330	0.006658	0.013116	0.006470	0.003394
	TH	0.127614	0.000487	0.000506	0.001617	0.001663	0.000451	0.003747	0.003294	0.001243	0.001592	0.001847
	R/L	0.075056	0.000830	0.002419	0.000773	0.001280	0.002142	0.001551	0.002429	0.000241	0.000378	0.001559
IE												
0.142857	LE	0.557143	0.008034	0.004066	0.004209	0.011348	0.004210	0.007660	0.004074	0.002062	0.002060	0.001448
	EH	0.122619	0.000798	0.000260	0.000471	0.000775	0.000430	0.000733	0.000297	0.000732	0.000308	0.000669
	ED	0.320238	0.001432	0.002520	0.002772	0.001016	0.002345	0.001129	0.001587	0.001606	0.000983	0.001375
		AHP weights	0.045579	0.060138	0.037659	0.060636	0.060474	0.065873	0.066230	0.045257	0.028057	0.042202

Table 7.4 AHP weights of 20 assets from cluster 2

Criteria	Subcriteria	Weights	Global weights									
			A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
IS												
0.571429	IN	0.277630	0.002956	0.001327	0.010248	0.005202	0.009420	0.011577	0.009713	0.006656	0.009231	0.001390
	SO	0.063455	0.000674	0.000331	0.003187	0.002486	0.002588	0.003100	0.002934	0.001555	0.001615	0.000298
	SA	0.555260	0.006125	0.002616	0.019643	0.017356	0.018703	0.019257	0.018516	0.014188	0.023982	0.002814
	SR	0.103655	0.001099	0.002724	0.002742	0.001281	0.005332	0.004783	0.003344	0.000629	0.000626	0.003430
IO												
0.285714	AG	0.249563	0.001249	0.002986	0.006659	0.003108	0.006124	0.005926	0.004652	0.005907	0.002706	0.000649
	DE	0.547768	0.002726	0.001364	0.011227	0.007851	0.010917	0.013063	0.009800	0.008367	0.010460	0.001306
	TH	0.127614	0.002799	0.001481	0.001448	0.001000	0.002956	0.000718	0.000514	0.000621	0.001317	0.001382
	R/L	0.075056	0.001546	0.001493	0.000766	0.000499	0.000394	0.000540	0.000614	0.000558	0.000743	0.003646
IE												
0.142857	LE	0.557143	0.001461	0.002822	0.003058	0.001576	0.006509	0.005893	0.005046	0.000924	0.001303	0.003313
	EH	0.122619	0.000323	0.000684	0.000678	0.000365	0.001483	0.001334	0.001091	0.001004	0.000181	0.000207
	ED	0.320238	0.004576	0.002256	0.002306	0.001727	0.001124	0.001145	0.001346	0.001240	0.001992	0.007973
		AHP weights	0.025534	0.020084	0.061962	0.042451	0.065550	0.067336	0.057572	0.041649	0.054156	0.026407

Criteria	Subcriteria	Weights	Global weights									
			A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
IS												
0.571429	IN	0.277630	0.008346	0.006833	0.006183	0.009136	0.010791	0.014060	0.009857	0.013117	0.005585	0.007017
	SO	0.063455	0.002316	0.002025	0.001405	0.001280	0.001480	0.001941	0.001980	0.002375	0.001187	0.001502
	SA	0.555260	0.023106	0.017411	0.012041	0.015949	0.021166	0.015043	0.021222	0.018269	0.012779	0.017107
	SR	0.103655	0.003214	0.002226	0.004295	0.004040	0.002400	0.003127	0.003292	0.003406	0.003637	0.003606
IO												
0.285714	AG	0.249563	0.003685	0.003623	0.004456	0.004117	0.002283	0.002439	0.000895	0.003077	0.004464	0.002299
	DE	0.547768	0.010203	0.007136	0.005708	0.007117	0.010867	0.009413	0.006799	0.011134	0.005009	0.006038
	TH	0.127614	0.005405	0.001385	0.001388	0.001307	0.001547	0.002840	0.001359	0.001446	0.002677	0.002869
	R/L	0.075056	0.000792	0.000854	0.000801	0.000514	0.000631	0.000822	0.001503	0.001507	0.001510	0.001711
IE												
0.142857	LE	0.557143	0.003530	0.003317	0.005468	0.005587	0.002828	0.004808	0.004523	0.004888	0.005994	0.006743
	EH	0.122619	0.001324	0.001350	0.001083	0.001247	0.000684	0.000568	0.001143	0.000810	0.001120	0.000839
	ED	0.320238	0.002517	0.002391	0.001539	0.001210	0.001632	0.002365	0.002315	0.002353	0.001424	0.002318
		AHP weights	0.064439	0.048549	0.044366	0.051504	0.056310	0.057426	0.054887	0.062381	0.045386	0.052051

Table 7.5 AHP weights of 20 assets from cluster 3

Criteria	Subcriteria	Weights	Global weights A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
IS												
0.571429	IN	0.277630	0.004710	0.015333	0.015333	0.008775	0.001328	0.014529	0.002449	0.008775	0.014396	0.004710
	SO	0.063455	0.001091	0.002122	0.002122	0.002138	0.000354	0.003833	0.000397	0.002444	0.003837	0.001091
	SA	0.555260	0.008836	0.025961	0.025489	0.016751	0.002720	0.030636	0.002935	0.017064	0.030707	0.008995
	SR	0.103655	0.001619	0.004997	0.005950	0.003081	0.000549	0.004896	0.000531	0.003081	0.005522	0.001619
IO												
0.285714	AG	0.249563	0.003436	0.001291	0.002004	0.003307	0.003276	0.000695	0.003204	0.003389	0.006904	0.007378
	DE	0.547768	0.007287	0.001919	0.002585	0.003914	0.006880	0.001368	0.007311	0.006942	0.013477	0.023180
	TH	0.127614	0.000833	0.001561	0.001517	0.001580	0.002606	0.004255	0.000431	0.000426	0.000447	0.000446
	R/L	0.075056	0.000857	0.000397	0.000447	0.000452	0.000872	0.000184	0.001545	0.000827	0.001613	0.002674
IE												
0.142857	LE	0.557143	0.001843	0.006557	0.003480	0.003480	0.002287	0.010906	0.000980	0.003591	0.000973	0.001898
	EH	0.122619	0.000460	0.000872	0.001590	0.001833	0.001865	0.001476	0.000220	0.000237	0.000228	0.000231
	ED	0.320238	0.001661	0.000890	0.003140	0.004831	0.000556	0.000939	0.000880	0.002878	0.000864	0.004054
		AHP weights	0.032631	0.061900	0.063657	0.050141	0.023294	0.073719	0.020882	0.049653	0.078968	0.056274

Criteria	Subcriteria	Weights	Global weights A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
IS												
0.571429	IN	0.277630	0.008775	0.014529	0.002072	0.004710	0.001695	0.002383	0.014529	0.008775	0.002065	0.008775
	SO	0.063455	0.003806	0.002159	0.000549	0.001081	0.000285	0.000567	0.003491	0.002189	0.000517	0.002189
	SA	0.555260	0.030394	0.024394	0.003493	0.008826	0.002804	0.004642	0.033053	0.018704	0.002779	0.018111
	SR	0.103655	0.004896	0.005495	0.000528	0.001619	0.000569	0.000844	0.005423	0.004415	0.000516	0.003081
IO												
0.285714	AG	0.249563	0.001115	0.001772	0.005911	0.003242	0.003469	0.002022	0.001809	0.003369	0.006598	0.007111
	DE	0.547768	0.001964	0.002737	0.013343	0.013076	0.007076	0.002780	0.002782	0.007038	0.013076	0.017770
	TH	0.127614	0.003467	0.001757	0.000854	0.000841	0.001523	0.002663	0.003211	0.002728	0.002571	0.002743
	R/L	0.075056	0.000267	0.000443	0.001628	0.001628	0.000866	0.001613	0.000443	0.000487	0.001670	0.002532
IE												
0.142857	LE	0.557143	0.000965	0.003544	0.003541	0.006261	0.001898	0.003591	0.006206	0.003591	0.006387	0.007613
	EH	0.122619	0.000481	0.000874	0.001418	0.000455	0.000486	0.000868	0.001661	0.000472	0.000907	0.000884
	ED	0.320238	0.004123	0.002499	0.004423	0.002924	0.004182	0.002922	0.000765	0.000816	0.001501	0.000900
		AHP weights	0.060254	0.060202	0.037761	0.044662	0.024853	0.024894	0.073373	0.052584	0.038588	0.071710

scale. That is, the local weights of the 20 assets with respect to each of the eleven subcriteria of suitability are calculated for each investor type.

After finding all the local weights, the global weights of each asset are determined by following what in terms of the AHP hierarchy may be regarded as a bottom-up process of successive multiplication. Illustratively speaking, the local weight of an asset in relation to a subcriterion is multiplied with the local weight of that subcriterion in relation to the respective parent criterion, which in turn, is multiplied with the local weight of the parent criterion in relation to the overall goal of suitability of assets. Thus, 11 global weights are obtained for each asset. The final AHP weight of suitability for each asset is then determined by adding all the global weights of the asset (refer to Tables 7.3-7.5).

7.5.3 Asset Allocation

The 20 financial assets from each cluster form the population from which we attempt to construct a portfolio comprising 8 assets. Table 7.6 provides the input data corresponding to expected short term return, expected long term return, risk and liquidity of assets from the three clusters. The main criteria of the problem instances solved are summarized in Table 7.7. The comparative values of the aspiration levels in Table 7.7 show diversity of the investor behavior. All the optimization models are coded and solved using LINGO 12.0.

Table 7.6 Input data of assets from cluster 1, cluster 2 & cluster 3

Clusters	Data	Assets									
		A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Cluster1											
	Expected short term return	0.02996	0.00329	0.17208	0.20376	0.21379	0.06883	0.08069	0.03861	0.19037	0.07952
	Expected long term return	0.11213	0.06742	0.15058	0.15292	0.28213	0.10498	0.11671	0.06898	0.20361	0.17279
	Risk	0.17948	0.14925	0.17243	0.16641	0.23336	0.16130	0.16756	0.13754	0.19171	0.19388
	Liquidity	0.00730	0.00067	0.00457	0.00650	0.02833	0.00750	0.00817	0.00567	0.00883	0.00757
Cluster2											
	Expected short term return	0.17377	0.16111	0.20249	0.09481	0.35012	0.28332	0.14264	0.17333	0.08511	0.19311
	Expected long term return	0.19278	0.21366	0.21711	0.19819	0.40086	0.30831	0.27858	0.27477	0.17133	0.17985
	Risk	0.13233	0.15847	0.17308	0.17779	0.26506	0.23237	0.21806	0.19649	0.17345	0.12945
	Liquidity	0.00040	0.00051	0.00070	0.00078	0.00228	0.00140	0.00508	0.00500	0.00078	0.00037
Cluster3											
	Expected short term return	0.10119	0.02790	0.13992	0.09751	0.22892	0.26715	0.14870	0.16130	0.12905	0.12591
	Expected long term return	0.13587	0.16622	0.14868	0.14382	0.17354	0.26879	0.10492	0.11692	0.10255	0.12328
	Risk	0.11661	0.15945	0.12344	0.12782	0.11865	0.19903	0.10628	0.11408	0.09853	0.08913
	Liquidity	0.00173	0.00480	0.00203	0.00430	0.00202	0.00640	0.00082	0.00080	0.00066	0.00075

Clusters	Data	Assets									
		A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
Cluster1											
	Expected short term return	0.01838	0.17869	0.15004	0.06406	0.05924	0.16050	0.15501	0.04218	0.03700	0.14039
	Expected long term return	0.09370	0.05809	0.10051	0.11579	0.09073	0.11533	0.06188	0.10039	0.04261	0.11500
	Risk	0.17002	0.13406	0.15926	0.14229	0.11245	0.16815	0.13386	0.23947	0.17112	0.16091
	Liquidity	0.00215	0.00173	0.00932	0.00892	0.00075	0.00333	0.00227	0.01018	0.00102	0.01038
Cluster2											
	Expected short term return	0.24057	0.15152	0.12064	0.32033	0.08501	0.06110	0.27495	0.18855	0.18030	0.39583
	Expected long term return	0.29892	0.29428	0.26969	0.34734	0.23776	0.21098	0.30107	0.29633	0.36700	0.30120
	Risk	0.19205	0.20298	0.16236	0.24781	0.20088	0.18182	0.17323	0.17112	0.21188	0.16652
	Liquidity	0.00040	0.00056	0.00113	0.00123	0.00103	0.00204	0.00072	0.00179	0.00064	0.00051
Cluster3											
	Expected short term return	0.15810	0.11333	0.10104	0.11453	0.12426	0.08967	0.24725	0.10843	0.26765	0.13041
	Expected long term return	0.19492	0.15927	0.12072	0.14921	0.12386	0.15267	0.23114	0.16003	0.16188	0.15436
	Risk	0.15032	0.13299	0.09515	0.11252	0.11217	0.09509	0.12940	0.13126	0.11311	0.10937
	Liquidity	0.00675	0.00458	0.00155	0.00512	0.00072	0.00160	0.00270	0.00212	0.00637	0.01042

Table 7.7 Main criteria of the problem instances solved

	Model P(7.4) for liquidity seekers	Model P(7.4) for return seekers	Model P(7.4) for safety seekers	Model P(7.5)
No. of assets	20	20	20	20
No. of criteria	5	5	5	5
Membership functions	nonlinear S-shape	nonlinear S-shape	nonlinear S-shape	nonlinear S-shape
Shape parameters	(i) $\alpha_{r12} = 60,\ \alpha_{r36} = 60, \alpha_w = 80,$ $\alpha_L = 600,\ \alpha_{w_{AHP}} = 600$ (ii) $\alpha_{r12} = 50,\ \alpha_{r36} = 50, \alpha_w = 100,$ $\alpha_L = 500,\ \alpha_{w_{AHP}} = 500$ (iii) $\alpha_{r12} = 40,\ \alpha_{r36} = 40, \alpha_w = 120,$ $\alpha_L = 400,\ \alpha_{w_{AHP}} = 400,$	(i) $\alpha_{r12} = 60,\ \alpha_{r36} = 60, \alpha_w = 80,$ $\alpha_L = 600,\ \alpha_{w_{AHP}} = 600$ (ii) $\alpha_{r12} = 50,\ \alpha_{r36} = 50, \alpha_w = 100,$ $\alpha_L = 500,\ \alpha_{w_{AHP}} = 500,$ (iii) $\alpha_{r12} = 40,\ \alpha_{r36} = 40,\ \alpha_w = 120,$ $\alpha_L = 400,\ \alpha_{w_{AHP}} = 400,$	(i) $\alpha_{r12} = 60,\ \alpha_{r36} = 60, \alpha_w = 80,$ $\alpha_L = 600,\ \alpha_{w_{AHP}} = 600$ (ii) $\alpha_{r12} = 50,\ \alpha_{r36} = 50, \alpha_w = 100,$ $\alpha_L = 500,\ \alpha_{w_{AHP}} = 500,$ (iii) $\alpha_{r12} = 40,\ \alpha_{r36} = 40,\ \alpha_w = 120,$ $\alpha_L = 400,\ \alpha_{w_{AHP}} = 400,$	$\alpha_{r12} = 60,\ \alpha_{r36} = 60,\ \alpha_w = 80$ $\alpha_L = 600,\ \alpha_{w_{AHP}} = 600$
Middle aspiration levels	$r_m^{12} = 0.15,\ r_m^{36} = 0.18,\ w_m = 0.215$ $L_m = 0.015,\ w_{AHP_m} = 0.035$	$r_m^{12} = 0.26,\ r_m^{36} = 0.325,\ w_m = 0.25$ $L_m = 0.002,\ w_{AHP_m} = 0.045$	$r_m^{12} = 0.185,\ r_m^{36} = 0.20,\ w_m = 0.175$ $L_m = 0.005,\ w_{AHP_m} = 0.06$	(i) $r_m^{12} = 0.15,\ r_m^{36} = 0.19,\ w_m = 0.215$ $L_m = 0.012,\ w_{AHP_m} = 0.035$ (ii) $r_m^{12} = 0.26,\ r_m^{36} = 0.325,\ w_m = 0.25$ $L_m = 0.002, w_{AHP_m} = 0.055$ (iii) $r_m^{12} = 0.205,\ r_m^{36} = 0.21,\ w_m = 0.19$ $L_m = 0.005,\ w_{AHP_m} = 0.068,$
No. of assets held in the portfolio	8	8	8	8
Criteria weights	—	—	—	(i) $\omega_1 = 0.25,\ \omega_2 = 0.2,\ \omega_3 = 0.1$ $\omega_4 = 0.3,\ \omega_5 = 0.15$ (ii) $\omega_1 = 0.3,\ \omega_2 = 0.25,\ \omega_3 = 0.15$ $\omega_4 = 0.1,\ \omega_5 = 0.2$ (iii) $\omega_1 = 0.18,\ \omega_2 = 0.12,\ \omega_3 = 0.35$ $\omega_4 = 0.10,\ \omega_5 = 0.25$

We now present the computational results.

- **Cluster 1 for liquidity seekers**

Corresponding to $r_m^{12} = 0.15$, $r_m^{36} = 0.18$, $w_m = 0.215$, $L_m = 0.015$, $w_{AHP_m} = 0.035$, $h = 8$, $l_i = 0.01$,, $u_i = 0.5$, $i = 1,2,\ldots,20$ and using the data from Tables 7.3, 7.6-7.7, we obtain portfolio selection by solving the problem P(7.4) formulated as follows:

$$\begin{aligned}
&\max \ \theta \\
&\text{subject to} \\
&1.79764x_1 + 0.19715x_2 + 10.32486x_3 + 12.22569x_4 + 12.82765x_5 + 4.12976x_6 \\
&+4.84157x_7 + 2.31676x_8 + 11.42234x_9 + 4.77094x_{10} + 1.10269x_{11} + 10.72153x_{12} \\
&+9.00203x_{13} + 3.84331x_{14} + 3.55451x_{15} + 9.63005x_{16} + 9.30053x_{17} + 2.53058x_{18} \\
&+2.21942x_{19} + 8.42338x_{20} - 9 \geq \theta\,, \\
&6.72784x_1 + 4.04520x_2 + 9.03493x_3 + 9.17531x_4 + 16.92760x_5 + 6.29862x_6 \\
&+7.00240x_7 + 4.13881x_8 + 12.21672x_9 + 10.36739x_{10} + 5.62217x_{11} + 3.48528x_{12} \\
&+6.03034x_{13} + 6.94756x_{14} + 5.44356x_{15} + 6.91998x_{16} + 3.71294x_{17} + 6.02335x_{18} \\
&+2.55661x_{19} + 6.89943x_{20} - 10.8 \geq \theta\,, \\
&34.47473x_1 + 47.28331x_2 + 42.01537x_3 + 26.44420x_4 + 17.77163x_5 + 22.50126x_6 \\
&+22.07219x_7 + 29.79401x_8 + 20.70612x_9 + 28.67455x_{10} + 27.34722x_{11} \\
&+36.08266x_{12} + 22.59514x_{13} + 36.38170x_{14} + 36.28428x_{15} + 39.52408x_{16} \\
&+39.73776x_{17} + 27.15403x_{18} + 16.83442x_{19} + 25.32134x_{20} - 21 \geq \theta\,, \\
&-14.35820x_1 - 11.93967x_2 - 13.79407x_3 - 13.31299x_4 - 18.66862x_5 - 12.90422x_6 \\
&-13.40499x_7 - 11.00324x_8 - 15.33695x_9 - 15.51005x_{10} - 13.60123x_{11} \\
&-10.72449x_{12} - 12.74118x_{13} - 11.38324x_{14} - 8.99547x_{15} - 13.45164x_{16} \\
&-10.70857x_{17} - 19.15735x_{18} - 13.68923x_{19} - 12.87272x_{20} + 17.2 \geq \theta\,, \\
&4.38x_1 + 0.4x_2 + 2.74x_3 + 3.9x_4 + 17x_5 + 4.5x_6 + 4.9x_7 + 3.4x_8 + 5.3x_9 + 4.54x_{10} \\
&+1.29x_{11} + 1.04x_{12} + 5.59x_{13} + 5.35x_{14} + 0.45x_{15} + 1.995x_{16} + 1.36x_{17} + 6.11x_{18} \\
&+0.612x_{19} + 6.23x_{20} - 9 \geq \theta\,, \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} \\
&+x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,, \\
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} \\
&+y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,, \\
&x_i - 0.01y_i \geq 0\,, \quad i = 1,2,\ldots,20\,, \\
&x_i - 0.5y_i \leq 0\,, \quad i = 1,2,\ldots,20\,, \\
&y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,20\,, \\
&x_i \geq 0\,, \quad i = 1,2,\ldots,20\,.
\end{aligned}$$

To check efficiency of the solution obtained, we use the two-phase approach and solve the problem P(7.6). If the investor is not satisfied with the portfolio obtained, more portfolios can be generated by varying the values of the shape parameters in the above problem. The computational results summarized in Table 7.8 are based on three different sets of values of the shape parameters.

Note that all the three solutions obtained are efficient, i.e., their criteria vector are nondominated. Table 7.9 presents proportions of the assets in the obtained portfolios.

- **Cluster 2 for return seekers**

Corresponding to $r_m^{12} = 0.26$, $r_m^{36} = 0.325$, $w_m = 0.25$, $L_m = 0.002$, $w_{AHP_m} = 0.045$, $h = 8$, $l_i = 0.01,$, $u_i = 0.5$, $i = 1, 2, \ldots, 20$ and using the data from Tables 7.4, 7.6-7.7, we obtain portfolio selection by solving the problem P(7.4) formulated as follows:

$$\begin{aligned}
&\max\ \theta\\
&\text{subject to}\\
&10.42630x_1 + 9.66667x_2 + 12.14964x_3 + 5.68882x_4 + 21.00698x_5 + 16.99939x_6\\
&+8.55833x_7 + 10.39949x_8 + 5.10631x_9 + 11.58647x_{10} + 14.43422x_{11} + 9.09091x_{12}\\
&+7.23849x_{13} + 19.21932x_{14} + 5.10069x_{15} + 3.66572x_{16} + 16.49695x_{17}\\
&+11.31296x_{18} + 10.81811x_{19} + 23.74959x_{20} - 15.6 \geq \theta\,,\\
&11.56676x_1 + 12.81936x_2 + 13.02644x_3 + 11.89147x_4 + 24.05182x_5 + 18.49870x_6\\
&+16.71450x_7 + 16.48592x_8 + 10.27970x_9 + 10.79095x_{10} + 17.93493x_{11}\\
&+17.65701x_{12} + 16.18129x_{13} + 20.84027x_{14} + 14.26543x_{15} + 12.65853x_{16}\\
&+18.06412x_{17} + 17.77963x_{18} + 22.02014x_{19} + 18.07171x_{20} - 19.5 \geq \theta\,,\\
&15.31975x_1 + 12.05122x_2 + 37.17770x_3 + 25.47086x_4 + 39.33009x_5 + 40.40148x_6\\
&+34.54299x_7 + 24.98953x_8 + 32.49344x_9 + 15.84419x_{10} + 38.66340x_{11}\\
&+29.12957x_{12} + 26.61943x_{13} + 30.90250x_{14} + 33.78613x_{15} + 34.45534x_{16}\\
&+32.93229x_{17} + 37.42836x_{18} + 27.23141x_{19} + 31.23033x_{20} - 27 \geq \theta\,,\\
&-10.58670x_1 - 12.67760x_2 - 13.84627x_3 - 14.22301x_4 - 21.20466x_5 - 18.58963x_6\\
&-17.44472x_7 - 15.71899x_8 - 13.87566x_9 - 10.35620x_{10} - 15.36419x_{11}\\
&-16.23817x_{12} - 12.98879x_{13} - 19.82475x_{14} - 16.07054x_{15} - 14.54573x_{16}\\
&-13.85837x_{17} - 13.68933x_{18} - 16.95061x_{19} - 13.32169x_{20} + 20 \geq \theta\,,\\
&0.241x_1 + 0.304x_2 + 0.42x_3 + 0.468x_4 + 1.37x_5 + 0.84x_6 + 3.05x_7 + 3x_8 + 0.465x_9\\
&+0.22x_{10} + 0.24x_{11} + 0.335x_{12} + 0.68x_{13} + 0.735x_{14} + 0.617x_{15} + 1.225x_{16}\\
&+0.43x_{17} + 1.075x_{18} + 0.385x_{19} + 0.307x_{20} - 1.2 \geq \theta\,,\\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}\\
&+x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,,\\
&y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10}\\
&+y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,,\\
&x_i - 0.01y_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,,\\
&x_i - 0.5y_i \leq 0\,, \quad i = 1, 2, \ldots, 20\,,\\
&y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 20\,,\\
&x_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,.
\end{aligned}$$

If the investor is not satisfied with the portfolio obtained, more portfolios can be generated by varying the values of the shape parameters in the above problem. The computational results summarized in Table 7.8 are based on three different sets of values of the shape parameters. Note that all the three solutions obtained are efficient. Table 7.10 presents proportions of the assets in the obtained portfolios.

- **Cluster 3 for safety seekers**

Corresponding to $r_m^{12} = 0.185$, $r_m^{36} = 0.20$, $w_m = 0.175$, $L_m = 0.005$, $w_{AHP_m} = 0.06$, $h = 8$, $l_i = 0.01$,, $u_i = 0.5$, $i = 1,2,\ldots,20$ and using the data from Tables 7.5, 7.6-7.7, we obtain portfolio selection by solving the problem P(7.4) formulated as follows:

$$\max\ \theta$$

subject to

$$6.07135x_1 + 1.67374x_2 + 8.39548x_3 + 5.85066x_4 + 13.73499x_5 + 16.02906x_6$$
$$+8.92199x_7 + 9.67795x_8 + 7.74328x_9 + 7.55484x_{10} + 9.48577x_{11} + 6.80007x_{12}$$
$$+6.06260x_{13} + 6.87204x_{14} + 7.45538x_{15} + 5.37990x_{16} + 14.83472x_{17}$$
$$+6.50549x_{18} + 16.05907x_{19} + 7.82519x_{20} - 11.1 \geq \theta\,,$$
$$8.15192x_1 + 9.97332x_2 + 8.92107x_3 + 8.62928x_4 + 10.41241x_5 + 16.12749x_6$$
$$+6.29500x_7 + 7.01501x_8 + 6.15299x_9 + 7.39683x_{10} + 11.69495x_{11} + 9.55592x_{12}$$
$$+7.24296x_{13} + 8.95255x_{14} + 7.43153x_{15} + 9.15999x_{16} + 13.86862x_{17}$$
$$+9.60144x_{18} + 9.71277x_{19} + 9.26141x_{20} - 12 \geq \theta\,,$$
$$19.57860x_1 + 37.140x_2 + 38.19420x_3 + 30.08460x_4 + 13.97640x_5 + 44.23140x_6$$
$$+12.52920x_7 + 29.79180x_8 + 47.38080x_9 + 33.76440x_{10} + 36.15240x_{11} + 36.12120x_{12}$$
$$+22.65660x_{13} + 26.79720x_{14} + 14.91180x_{15} + 14.93640x_{16} + 44.02380x_{17}$$
$$+31.55040x_{18} + 23.15280x_{19} + 43.0260x_{20} - 36 \geq \theta\,,$$
$$-9.32879x_1 - 12.75605x_2 - 9.87534x_3 - 10.22537x_4 - 9.49203x_5 - 15.92201x_6$$
$$-8.50213x_7 - 9.12645x_8 - 7.88276x_9 - 7.13068x_{10} - 12.02566x_{11} - 10.63882x_{12}$$
$$-7.61216x_{13} - 9.00134x_{14} - 8.97327x_{15} - 7.60692x_{16} - 10.35152x_{17}$$
$$-10.50074x_{18} - 9.04904x_{19} - 8.74996x_{20} + 14 \geq \theta\,,$$
$$0.241x_1 + 0.304x_2 + 0.42x_3 + 0.468x_4 + 1.37x_5 + 0.84x_6$$
$$+3.05x_7 + 3x_8 + 0.465x_9 + 0.22x_{10} + 0.24x_{11} + 0.335x_{12}$$
$$+0.68x_{13} + 0.735x_{14} + 0.617x_{15} + 1.225x_{16} + 0.43x_{17}$$
$$+1.075x_{18} + 0.385x_{19} + 0.307x_{20} - 3 \geq \theta\,,$$
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}$$
$$+x_{11} + x_{12} + x_{13} + x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,,$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10}$$
$$+y_{11} + y_{12} + y_{13} + y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,,$$
$$x_i - 0.01y_i \geq 0\,, \quad i = 1,2,\ldots,20\,,$$
$$x_i - 0.5y_i \leq 0\,, \quad i = 1,2,\ldots,20\,,$$
$$y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,20\,,$$
$$x_i \geq 0\,, \quad i = 1,2,\ldots,20\,.$$

Table 7.8 Summary results of portfolio selection for cluster 1, cluster 2 & cluster 3

Clusters	Shape parameters & variables							Risk	Expected return		Liquidity	AHP weighted
	η	θ	α_{r36}	α_{r12}	α_w	α_L	$\alpha_{w_{AHP}}$		Long term	Short term		score
Cluster 1												
	0.66342	0.67857	60	60	80	600	600	0.19781	0.19131	0.16618	0.01613	0.03625
	0.63732	0.56375	50	50	100	500	500	0.19854	0.19128	0.16536	0.01613	0.03627
	0.57566	0.30498	40	40	120	400	400	0.19867	0.18763	0.16190	0.01577	0.03658
Cluster 2												
	0.65144	0.62538	60	60	80	600	600	0.23085	0.33542	0.27042	0.00304	0.05348
	0.62477	0.50985	50	50	100	500	500	0.23117	0.33520	0.27020	0.00302	0.05361
	0.59840	0.39880	40	40	120	400	400	0.23149	0.33497	0.26997	0.00300	0.05373
Cluster 3												
	0.73947	1.04322	60	60	80	600	600	0.15939	0.21739	0.20239	0.00674	0.06683
	0.69702	0.83315	50	50	100	500	500	0.15834	0.21666	0.20166	0.00667	0.06669
	0.65752	0.65226	40	40	120	400	400	0.15781	0.21631	0.20131	0.00663	0.06662

Table 7.9 The proportions of the assets in the obtained portfolios for cluster 1

Shape parameters					Allocation									
α_{r36}	α_{r12}	α_w	α_L	$\alpha_{w_{AHP}}$	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
60	60	80	600	600	0	0	0	0	0.35	0.025	0.02	0	0.19962	0.026
50	50	100	500	500	0	0	0	0	0.35	0.025	0.02	0	0.20072	0.026
40	40	120	400	400	0	0	0	0	0.33	0.025	0.02	0	0.20055	0.026
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0	0	0.023	0	0	0	0	0.05638	0	0.30
					0	0	0.023	0	0	0	0	0.06528	0	0.29
					0	0	0.023	0	0	0	0	0.08545	0	0.29

Table 7.10 The proportions of the assets in the obtained portfolios for cluster 2

Shape parameters					Allocation									
$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	$\alpha_{w_{AHP}}$	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
60	60	80	600	600	0	0	0	0	0.35	0.025	0.02986	0.35909	0	0
50	50	100	500	500	0	0	0	0	0.34	0.025	0.04430	0.34114	0	0
40	40	120	400	400	0	0	0	0	0.33	0.025	0.05873	0.32322	0	0
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0	0	0	0.18805	0	0	0.023	0	0.01	0.015
					0	0	0	0.20156	0	0	0.023	0	0.01	0.015
					0	0	0	0.21505	0	0	0.023	0	0.01	0.015

Table 7.11 The proportions of the assets in the obtained portfolios for cluster 3

Shape parameters					Allocation									
$\alpha_{r^{36}}$	$\alpha_{r^{12}}$	α_w	α_L	$\alpha_{w_{AHP}}$	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
60	60	80	600	600	0	0	0	0	0.019	0.40	0	0	0	0
50	50	100	500	500	0	0	0	0	0.019	0.38	0	0	0	0
40	40	120	400	400	0	0	0	0	0.019	0.37	0	0	0	0
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0.29992	0.03	0	0	0	0.016	0.05902	0	0.01	0.16606
					0.31079	0.03	0	0	0	0.016	0.07365	0	0.01	0.16056
					0.31630	0.03	0	0	0	0.016	0.0810	0	0.01	0.15770

Table 7.12 Summary result of portfolio selection for cluster 3 (improved solution)

α_{r36}	α_{r12}	α_w	α_L	$\alpha_{w_{AHP}}$	Risk	Expected return		Liquidity	AHP weighted score
						Long term	Short term		
40	40	120	400	400	0.15773	0.21631	0.20140	0.00663	0.06665

Table 7.13 The proportions of the assets in the obtained portfolio (cluster 3) corresponding to improved solution

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Portfolio	0	0	0	0	0.019	0.37	0	0	0	0
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0.31362	0.03	0	0	0	0.016	0.08241	0	0.01	0.15897

The computational results are summarized in Table 7.8. Table 7.11 presents proportions of the assets in the obtained portfolios. Note that unlike the situation in cluster 1 and cluster 2, in this cluster, the solution obtained corresponding to the shape parameters $\alpha_{r12} = 40$, $\alpha_{r36} = 40$, $\alpha_w = 120$, $\alpha_L = 400$, $\alpha_{w_{AHP}} = 400$ is not efficient. In this case, the recourse to the two-phase approach produces the efficient solution listed in Tables 7.12-7.13. It may be noted that the solution criteria vector $(0.15773, 0.21631, 0.20140, 0.00663, 0.06665)$ of the Table 7.12 dominates the solution criteria vector $(0.15781, 0.21631, 0.20131, 0.00663, 0.06662)$ of the Table 7.8 from cluster 3 corresponding to the shape parameters $\alpha_{r12} = 40$, $\alpha_{r36} = 40$, $\alpha_w = 120$, $\alpha_L = 400$, $\alpha_{w_{AHP}} = 400$.

A comparison of the solutions for the three clusters listed in Table 7.8 highlights that if investors are liquidity seekers, they will obtain a higher level of liquidity in comparison to return seekers and safety seekers, but that supposes a medium risk level. If investors are return seekers, they will obtain a higher level of expected return in comparison to liquidity seekers and safety seekers, but that supposes a higher risk level. If investors are safety seekers, they will obtain a lower level of risk in comparison to liquidity seekers and return seekers, but that supposes accepting medium level of expected return.

Apart from optimizing the return, risk and liquidity objectives in the obtained portfolios, we also analyze how the obtained portfolios perform on the suitability considerations and overall satisfaction of investor preferences. The normalized AHP weighted score of suitability of the portfolio lies between 0 and 1. It may be recalled that the overall suitability score captures suitability of the portfolio on three criteria and eleven subcriteria. Further to this, and in view of the multiobjective portfolio selection problem, it is quite possible that an asset that has a high AHP weight does not figure in the optimal portfolio because it may not adequately contribute towards the attainment of the other objectives. For example, asset A2 from cluster 1 has the highest AHP weight but it does not figure in the obtained portfolios as it is not performing well on the other objectives. Therefore, just as an optimal portfolio may not comprise of only those assets that yield highest return, liquidity or lowest risk, likewise, it does not necessarily include the assets with the highest AHP weight. What matters is the 'portfolio effect' of the individual assets, be it return, be it liquidity, be it risk or be it suitability.

Next, we present computational results considering individual preferences within a given investor type.

• Individual preferences among liquidity seekers
We consider the following weights of the fuzzy goals of expected short term return (ω_1), expected long term return (ω_2), risk (ω_3), liquidity (ω_4) and AHP weighted score (ω_5): $\omega_1 = 0.25$, $\omega_2 = 0.2$, $\omega_3 = 0.1$, $\omega_4 = 0.3$, $\omega_5 = 0.15$. Corresponding to $r_m^{12} = 0.15$, $r_m^{36} = 0.19$, $w_m = 0.215$, $L_m = 0.012$, $w_{AHP_m} = 0.035$, $h = 8$, $l_i = 0.01,$, and $u_i = 0.5,\ i = 1, 2, \ldots, 20$, we obtain portfolio selection by solving the problem P(7.5). The efficiency of the solution is verified by solving the problem P(7.7) in the second-phase. The corresponding computational results are listed in Tables 7.14-7.15. The achievement levels of the various membership functions are $\eta_1 = 0.86063$, $\eta_2 = 0.81308$, $\eta_3 = 0.73167$, $\eta_4 = 0.87916$, $\eta_5 = 0.75755$ which are consistent with the investor preferences, i.e., ($\eta_4 > \eta_1 > \eta_2 > \eta_5 > \eta_3$) agrees with ($\omega_4 > \omega_1 > \omega_2 > \omega_5 > \omega_3$).

• Individual preferences among return seekers
Here, we consider the weights as $\omega_1 = 0.3$, $\omega_2 = 0.25$, $\omega_3 = 0.15$, $\omega_4 = 0.10$, $\omega_5 = 0.20$. By taking $r_m^{12} = 0.26$, $r_m^{36} = 0.325$, $w_m = 0.25$, $L_m = 0.002$, $w_{AHP_m} = 0.055$, $h = 8$, $l_i = 0.01,$, and $u_i = 0.5,\ i = 1, 2, \ldots, 20$, we obtain portfolio selection by solving the problem P(7.5). The solution is verified for efficiency. The corresponding computational results are listed in Tables 7.14-7.15. The achievement levels of the various membership functions are $\eta_1 = 0.97465$, $\eta_2 = 0.88775$, $\eta_3 = 0.57460$, $\eta_4 = 0.46035$, $\eta_5 = 0.72716$ which are consistent with the investor preferences.

• Individual preferences among safety seekers
As performed above in cluster 1 and cluster 2, corresponding to the weights $\omega_1 = 0.18$, $\omega_2 = 0.12$, $\omega_3 = 0.35$, $\omega_4 = 0.10$, $\omega_5 = 0.25$ and $r_m^{12} = 0.205$, $r_m^{36} = 0.21$, $w_m = 0.19$, $L_m = 0.005$, $w_{AHP_m} = 0.068$, $h = 8$, $l_i = 0.01,$, $u_i = 0.5,\ i = 1, 2, \ldots, 20$, we obtain portfolio selection by solving the problem P(7.5). The solution is found to be efficient. The corresponding computational results are listed in Tables 7.14-7.15. The achievement levels of the various membership functions are $\eta_1 = 0.88969$, $\eta_2 = 0.81399$, $\eta_3 = 0.93570$, $\eta_4 = 0.42061$, $\eta_5 = 0.92636$ which are consistent with the investor preferences.

Table 7.14 Summary results of portfolio selection incorporating investor preferences for cluster 1, cluster 2 & cluster 3

Clusters	Shape parameters					Risk	Expected return		Liquidity	AHP weighted
	α_{r36}	α_{r12}	α_w	α_L	$\alpha_{w_{AHP}}$		Long term	Short term		score
Cluster 1										
	60	60	80	600	600	0.20246	0.21450	0.18034	0.01531	0.03690
Cluster 2										
	60	60	80	600	600	0.24624	0.35947	0.320825	0.00174	0.05663
Cluster 3										
	60	60	80	600	600	0.15653	0.23460	0.23979	0.00447	0.07222

Table 7.15 The proportions of the assets in the obtained portfolios incorporating investor preferences

Clusters	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Cluster 1	0.10	0	0.02	0.025	0.35	0	0	0	0.451	0
Cluster 2	0	0	0	0	0.35	0.17498	0.02	0	0	0
Cluster 3	0	0	0.02	0	0	0.40	0	0	0.035	0
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0	0	0.023	0	0	0.016	0	0	0	0.015
	0.04	0	0	0.367	0	0	0.023	0	0.01	0.015
	0.04	0.03	0	0	0	0	0.45	0	0.01	0.015

The foregoing analysis of the various decision making situations from the stand point of investor preferences demonstrates that the portfolio selection models discussed in this chapter discriminate not only between investor types but also among investors in a given type. Thus, it is possible to construct efficient portfolios with reference to the diversity of investor preferences whether considered at the level of investor type or at the level of an individual within a given type.

7.6 Comments

In this chapter, we have presented the following facts:

- A hybrid approach that integrates behavior survey, cluster analysis, AHP and fuzzy mathematical programming has been discussed to study the portfolio selection problem.
- Based on a survey data in respect of investors demographic, socio-cultural, economic and psychographic profiles, investors have been categorized as return seekers, safety seekers and liquidity seekers.
- Cluster analysis has been introduced to categorize the chosen sample of financial assets into three clusters-cluster 1: liquid assets; cluster 2: high-yield assets; and cluster 3: less risky assets.
- Using AHP approach, the suitability of the assets from a particular cluster has been measured for a given investor type.
- The convergence of the dual goals of suitability and optimality in portfolio selection has been introduced.
- Recognizing that financial investment involves multiple criteria decision making in an environment that befits more fuzzy approximation than deterministic formulation, the transformation of the semi-absolute deviation

portfolio selection model into a fuzzy model using nonlinear S-shape membership functions has been discussed.
- The computational results based on real-world data for each of the three clusters have been provided to demonstrate the effectiveness of the portfolio selection models. Further, the efficiency of the obtained solutions has been verified using the two-phase approach.
- The advantage of the portfolio selection models have been shown under the situation that if investors are not satisfied with any of the portfolios, more portfolios can be generated by varying the values of the shape parameters.
- Moreover, it has been shown that the fuzzy portfolio selection models are capable of yielding suitable and optimal portfolios not only for each investor type but can also accommodate individual preferences within a given type.

Chapter 8
Suitability Considerations in Multi-criteria Fuzzy Portfolio Optimization-II

Abstract. In this chapter, we present an approach based on AHP and fuzzy multiobjective programming (FMOP) to attain the convergence of suitability and optimality in portfolio selection. We use a typology of investors with a view to discriminate among investors types and asset clusters categorized on the basis of three evaluation indices. The local weights (performance scores) of each asset within a cluster with respect to the four key criteria, namely, return, risk, liquidity and suitability are calculated using AHP. These weights are used as coefficients of the objective functions corresponding to the four criteria in the multiobjective programming model. The multiobjective programming model is transformed into a weighted additive model using the weights (relative importance) of the four key criteria that directly influence the asset allocation decision. These criteria weights are also calculated using AHP. To improve portfolio performance on individual objective(s) as per investor preferences, we use an interactive fuzzy programming approach.

8.1 AHP Model for Suitability and Optimality Considerations

As discussed in the previous chapter, we use the following triadic typology of investor behavior: return seekers, safety seekers and liquidity seekers. Further, we use the following three clusters of assets as obtained in previous chapter.

(i) ***Cluster 1: liquid assets***
Assets in Cluster 1 are categorized as liquid assets, as mean value for liquidity is the highest in this cluster. This cluster is typified by low but widely varying returns.

(ii) ***Cluster 2: high-yield assets***
Assets in Cluster 2 are categorized as high-yielding ones, since they have rather high returns. On the expected lines of return/risk relationship, these assets also show high standard deviation. Although, investors may gain from

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the high returns, they also have to endure the high risk. However, these assets have low liquidity amongst all the clusters indicating that high-yielding investment involves a longer time horizon.

(iii) ***Cluster 3: less risky assets***
Assets in Cluster 3 are categorized as less risky assets, since compared to other clusters, these assets manifest the lowest standard deviation for the cluster. The return is not high but medium. The liquidity is medium too.

These asset clusters have a *prima facie* suitability for the above stated investor types. We calculate the local weights (performance scores) of each asset within a cluster with respect to key asset allocation criteria and the weights (relative importance) of the key criteria when making the asset allocation decision using AHP. The AHP model used here comprises five levels of hierarchy. Level 1 represents the overall goal, i.e. Asset Allocation. Level 2 represents the key criteria (Return, Risk, Liquidity and Suitability) that directly influence the goal. At level 3, Return criterion is broken into Short Term Return, Long Term Return and Forecasted Return; Risk criterion is broken into Standard Deviation, Risk Tolerance and Microeconomic Risk; Suitability criterion is broken into Income and Savings, Investment Objectives and Investing Experience. At level 4, suitability subcriteria are further broken into 11 subcriteria that may affect the choice of assets. At the bottom level of the hierarchy, the alternatives (i.e., assets) are listed (please refer to Fig. 8.1 for complete structural hierarchy).

Note that to determine the local weights of the assets with respect to the quantitative measures of performance, namely, short term return, long term return, risk (standard deviation) and liquidity, we have relied on actual data, that is, the past performance of the assets. The question for pairwise comparison of quantitative criteria can be considered as:

'*Of two elements i and j, how many times i is preferred to j*'

If the values for the alternatives i and j are, respectively, w_i and w_j, the preference of the alternative i to j is equal to w_i/w_j. Therefore, the pairwise comparison matrix is

$$\begin{pmatrix} w_1/w_1 & w_1/w_2 & \dots & w_1/w_n \\ w_2/w_1 & w_2/w_2 & \dots & w_2/w_n \\ & & \dots & \\ w_n/w_1 & w_n/w_2 & \dots & w_n/w_n \end{pmatrix}.$$

As this matrix is consistent, the weight of i-th element is its relative normalized amount, i.e., $\dfrac{w_i}{\sum_{i=1}^{n} w_i}$.

The priority of the alternative i to j for negative criterion, such as risk, is equal to w_j/w_i. The pairwise comparison matrix is therefore

$$\begin{pmatrix} w_1/w_1 & w_2/w_1 & \dots & w_n/w_1 \\ w_1/w_2 & w_2/w_2 & \dots & w_n/w_2 \\ & & \dots & \\ w_1/w_n & w_2/w_n & \dots & w_n/w_n \end{pmatrix}.$$

The above matrix is consistent [104] and the weight of the i-th element (for negative criteria) is $\dfrac{1/w_i}{\sum_{i=1}^{n} 1/w_i}$.

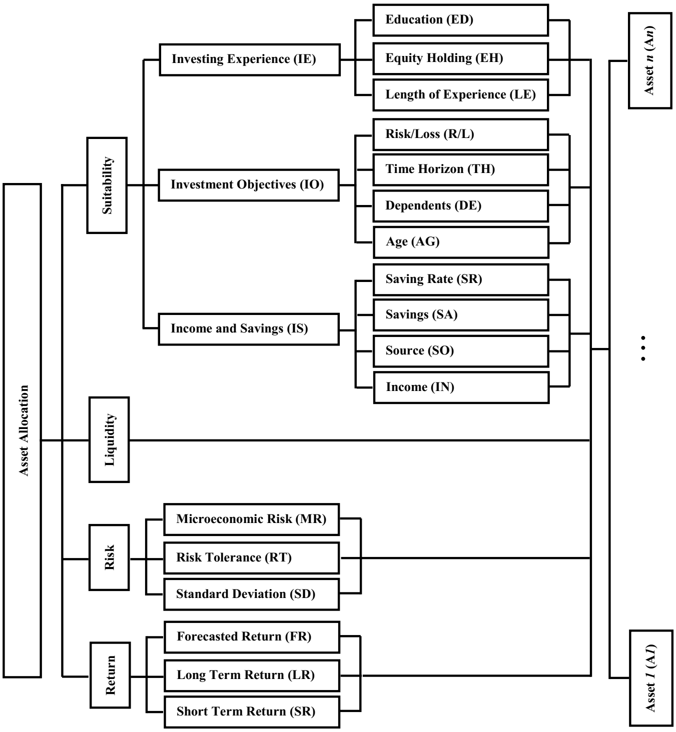

Fig. 8.1 Structural hierarchy for asset allocation

8.2 Fuzzy Multiobjective Portfolio Selection Model

Assume that investors allocate their wealth among n assets. We introduce some notation as follows:

8.2.1 Notation

r_i: the AHP-local weight of the i-th asset with respect to return,

b_i: the AHP-local weight of the i-th asset with respect to risk,

l_i: the AHP-local weight of the i-th asset with respect to liquidity,

s_i: the AHP-local weight of the i-th asset with respect to suitability,

θ_1: the AHP-weight of the return criterion,

θ_2: the AHP-weight of the risk criterion,

θ_3: the AHP-weight of the liquidity criterion,

θ_4: the AHP-weight of the suitability criterion,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

h: the number of assets held in the portfolio.

We consider the following objective functions and constraints in the multiobjective portfolio selection problem.

8.2.2 Objective Functions

AHP-Weighted Score with Respect to Return

The AHP-weighted score with respect to return of the portfolio is expressed as

$$f_1(x) = \sum_{i=1}^{n} r_i x_i.$$

AHP-Weighted Score with Respect to Risk

The AHP-weighted score with respect to risk of the portfolio is expressed as

$$f_2(x) = \sum_{i=1}^{n} b_i x_i.$$

AHP-Weighted Score with Respect to Liquidity

The AHP-weighted score with respect to liquidity of the portfolio is expressed as

$$f_3(x) = \sum_{i=1}^{n} l_i x_i .$$

AHP-Weighted Score with Respect to Suitability
The AHP-weighted score with respect to suitability of the portfolio is expressed as

$$f_4(x) = \sum_{i=1}^{n} s_i x_i .$$

8.2.3 Constraints

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1 .$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \le u_i y_i , \quad i = 1, 2, \ldots, n .$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \ge l_i y_i , \quad i = 1, 2, \ldots, n .$$

Number of assets held in a portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h$$

No short selling of assets is expressed as

$$x_i \ge 0 , \quad i = 1, 2, \ldots, n .$$

8.2.4 The Decision Problem

The multiobjective mixed integer linear programming model of portfolio selection using AHP-local weights of suitability and optimality is formulated as

$$\mathbf{P(8.1)}\quad \max f_1(x) = \sum_{i=1}^{n} r_i x_i$$

$$\max f_2(x) = \sum_{i=1}^{n} b_i x_i$$

$$\max f_3(x) = \sum_{i=1}^{n} l_i x_i$$

$$\max f_4(x) = \sum_{i=1}^{n} s_i x_i$$

subject to

$$\sum_{i=1}^{n} x_i = 1\,, \tag{8.1}$$

$$\sum_{i=1}^{n} y_i = h\,, \tag{8.2}$$

$$x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n\,, \tag{8.3}$$

$$x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n\,, \tag{8.4}$$

$$x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,, \tag{8.5}$$

$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n. \tag{8.6}$$

8.3 Solution Methodology

To handle the multiobjective programming model P(8.1), we present an interactive fuzzy programming approach [1]. The solution methodology of the interactive fuzzy programming approach consists of the following steps:

Step 1: Construct the model P(8.1) using the AHP-local weights (performance scores) of each asset within a cluster with respect to the four key criteria-return, risk, liquidity and suitability.

Step 2: Solve the problem P(8.1) as a single-objective problem in respect each criterion. Mathematically, we solve the following problems:

(*i*) $\max f_1(x)$ subject to constraints (8.1)-(8.6).

(*ii*) $\max f_2(x)$ subject to constraints (8.1)-(8.6).

(*iii*) $\max f_3(x)$ subject to constraints (8.1)-(8.6).

(*iv*) $\max f_4(x)$ subject to constraints (8.1)-(8.6).

Let x^1, x^2, x^3 and x^4 denote the optimal solutions obtained by solving the above defined single-objective problems. If all the solutions, i.e., $x^1 = x^2 = x^3 = x^4 = (x_1, x_2, \ldots, x_n)$ are same, we obtain an efficient (preferred compromise) solution and stop; otherwise, go to Step 3.

Step 3: Evaluate all the objective functions at the obtained solutions. Determine the worst lower bound (f_1^L) and best upper bound (f_1^R) for return criterion; the worst lower bound (f_2^L) and best upper bound (f_2^R) for risk criterion; the worst lower bound (f_3^L) and best upper bound (f_3^R) for liquidity criterion; and, the worst lower bound (f_4^L) and best upper bound (f_4^R) for suitability criterion. We obtain these bounds as

$$\begin{aligned}
f_1^R &= \max\{f_1(x^1), f_1(x^2), f_1(x^3), f_1(x^4)\},\\
f_1^L &= \min\{f_1(x^1), f_1(x^2), f_1(x^3), f_1(x^4)\},\\
f_2^R &= \max\{f_2(x^1), f_2(x^2), f_2(x^3), f_2(x^4)\},\\
f_2^L &= \min\{f_2(x^1), f_2(x^2), f_2(x^3), f_2(x^4)\},\\
f_3^R &= \max\{f_3(x^1), f_3(x^2), f_3(x^3), f_3(x^4)\},\\
f_3^L &= \min\{f_3(x^1), f_3(x^2), f_3(x^3), f_3(x^4)\},\\
f_4^R &= \max\{f_4(x^1), f_4(x^2), f_4(x^3), f_4(x^4)\},\\
f_4^L &= \min\{f_4(x^1), f_4(x^2), f_4(x^3), f_4(x^4)\}.
\end{aligned}$$

Step 4: Define the linear membership functions for return, risk, liquidity and suitability criteria as follows:

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq f_1^R, \\ \frac{f_1(x)-f_1^L}{f_1^R-f_1^L}, & \text{if } f_1^L \leq f_1(x) \leq f_1^R, \\ 0, & \text{if } f_1(x) \leq f_1^L, \end{cases}$$

where $\mu_{f_1}(x)$ denotes the satisfaction degree of return criterion for a given solution x.

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \geq f_2^R, \\ \frac{f_2(x)-f_2^L}{f_2^R-f_2^L}, & \text{if } f_2^L \leq f_2(x) \leq f_2^R, \\ 0, & \text{if } f_2(x) \leq f_2^L, \end{cases}$$

where $\mu_{f_2}(x)$ denotes the satisfaction degree of risk criterion for a given solution x.

$$\mu_{f_3}(x) = \begin{cases} 1, & \text{if } f_3(x) \geq f_3^R, \\ \frac{f_3(x)-f_3^L}{f_3^R-f_3^L}, & \text{if } f_3^L \leq f_3(x) \leq f_3^R, \\ 0, & \text{if } f_3(x) \leq f_3^L, \end{cases}$$

where $\mu_{f_3}(x)$ denotes the satisfaction degree of liquidity criterion for a given solution x.

$$\mu_{f_4}(x) = \begin{cases} 1, & \text{if } f_4(x) \geq f_4^R\,, \\ \frac{f_4(x)-f_4^L}{f_4^R-f_4^L}, & \text{if } f_4^L \leq f_4(x) \leq f_4^R\,, \\ 0, & \text{if } f_4(x) \leq f_4^L\,, \end{cases}$$

where $\mu_{f_4}(x)$ denotes the satisfaction degree of suitability criterion for a given solution x.

Step 5: Convert multiobjective problem P(8.1) into a single-objective problem P(8.2) using 'weighted additive approach' based on the AHP-weights in respect of each criterion as follows:

$$\begin{aligned} \textbf{P(8.2)}\quad & \max\ \theta_1\alpha_1 + \theta_2\alpha_2 + \theta_3\alpha_3 + \theta_4\alpha_4 \\ & \text{subject to} \\ & \alpha_1 \leq \mu_{f_1}(x)\,, \\ & \alpha_2 \leq \mu_{f_2}(x)\,, \\ & \alpha_3 \leq \mu_{f_3}(x)\,, \\ & \alpha_4 \leq \mu_{f_4}(x)\,, \\ & 0 \leq \alpha_1 \leq 1\,, \\ & 0 \leq \alpha_2 \leq 1\,, \\ & 0 \leq \alpha_3 \leq 1\,, \\ & 0 \leq \alpha_4 \leq 1\,, \\ & \text{and Constraints (8.1)} - \text{(8.6)}\,, \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the lower bounds on the satisfaction levels corresponding to return, risk, liquidity and suitability criteria, respectively.

Step 6: Solve the single-objective problem P(8.2) using the AHP-weights of the key criteria of asset allocation that reflect relative importance of each criterion in portfolio selection. Present the solution to the investor. If the investor is satisfied with the obtained preferred compromise solution, then stop and select the current solution as the final decision; otherwise, evaluate the objective function(s) at the obtained solution in which the investor wishes improvement. Compare the lower bound of objective function(s) with the obtained new value(s). If the new value is higher than the current lower bound, consider it as a new lower bound. If there are no changes in current lower bound(s) of the desired objective function(s) then stop; otherwise, go to Step 4.

The solution process terminates when the investor accepts the obtained solution and consider it as the preferred compromise solution which is in fact a compromise feasible solution that meets the investor preferences.

8.4 Numerical Illustration

To demonstrate the applicability of the portfolio selection approach, we present an empirical study done using a data set of daily closing prices in respect of 150 assets listed on NSE, Mumbai, India. As demonstrated in the previous chapter these assets have been categorized into three clusters, namely, cluster 1: liquid assets (56 assets), cluster 2: high-yield assets (51 assets) and cluster 3: less risky assets (43 assets).

8.4.1 Calculation of AHP Weighted Scores

Here, we refer to Fig. 8.1 for complete hierarchy. Note that for the empirical testing of the AHP model, we have randomly chosen 20 assets from each cluster to ensure consistency in the application of the methodology. For the data in respect of pair-wise comparisons involving qualitative measures (criteria and subcriteria), we use inputs from investors that are based on the Saaty's verbal scale [104]. For the data in respect of pair-wise comparisons involving quantitative measures, the real quantitative data listed in Table 8.1 is used for the three clusters.

The pair-wise comparison process used in this study moves from the top of the hierarchy down. In Fig. 8.1, the four key criteria are first compared with respect to the overall goal. The three subcriteria beneath the return criterion are then pair-wise compared. Similarly, the subcriteria beneath the risk criterion and the suitability criterion are pair-wise compared. The various subcriteria beneath each of the suitability subcriteria are also pair-wise compared. Finally, pair-wise comparisons of asset 1 through asset 20 are made with respect to various subcriteria and liquidity criterion. The local weights (performance scores) assigned to an asset with respect to the four key criteria are found by tracing the paths that lead from the respective key criterion down to the asset, multiplying the weights of the branches in the path to determine the weight of the path, and adding these path weights together. The weights (relative importance) of the four key criteria with respect to the overall goal are obtained from pair-wise comparisons with respect to the goal. For the sake of brevity, we have not included all the pair-wise comparison details for calculating the AHP-weights of the criteria, subcriteria, and alternatives. However, weights used in calculation of the performance scores of the assets for all the three clusters are provided in the Tables 8.2-8.4. Table 8.5 presents the local weights of the 20 assets for cluster 1, cluster 2 and cluster 3 and also the criteria weights.

Table 8.1 Input data of assets for quantitative criteria from cluster 1, cluster 2 & cluster 3

Cluster	Criteria	Assets									
		A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Cluster1											
	Expected short term return	0.02996	0.00329	0.17208	0.20376	0.21379	0.06883	0.08069	0.03861	0.19037	0.07952
	Expected long term return	0.11213	0.06742	0.15058	0.15292	0.28213	0.10498	0.11671	0.06898	0.20361	0.17279
	Risk (standard deviation)	0.44464	0.39196	0.43562	0.44955	0.66807	0.43017	0.46513	0.40143	0.53714	0.48292
	Liquidity	0.00611	0.00027	0.00359	0.00478	0.01378	0.00570	0.00937	0.00410	0.01149	0.00634
Cluster2											
	Expected short term return	0.17377	0.16111	0.20249	0.09481	0.35012	0.28332	0.14264	0.17333	0.08511	0.19311
	Expected long term return	0.19278	0.21366	0.21711	0.19819	0.40086	0.30831	0.27858	0.27477	0.17133	0.17985
	Risk (standard deviation)	0.37430	0.41053	0.45425	0.52204	0.65059	0.61920	0.60419	0.58206	0.45553	0.36399
	Liquidity	0.00032	0.00126	0.00124	0.00134	0.00352	0.00241	0.00631	0.00415	0.00180	0.00060
Cluster3											
	Expected short term return	0.10119	0.02790	0.13992	0.09751	0.22892	0.26715	0.14870	0.16130	0.12905	0.12591
	Expected long term return	0.13587	0.16622	0.14868	0.14382	0.17354	0.26879	0.10492	0.11692	0.10255	0.12328
	Risk (standard deviation)	0.29466	0.38806	0.31859	0.31315	0.28442	0.55536	0.27518	0.28486	0.25876	0.24466
	Liquidity	0.00088	0.00504	0.00201	0.00251	0.00164	0.01234	0.00059	0.00221	0.00045	0.00092

Cluster	Criteria	Assets									
		A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
Cluster1											
	Expected short term return	0.01838	0.17869	0.15004	0.06406	0.05924	0.16050	0.15501	0.04218	0.03700	0.14039
	Expected long term return	0.09370	0.05809	0.10051	0.11579	0.09073	0.11533	0.06188	0.10039	0.04261	0.11500
	Risk (standard deviation)	0.44688	0.32967	0.45308	0.38205	0.32435	0.39966	0.32673	0.69076	0.49083	0.39374
	Liquidity	0.00185	0.00164	0.00570	0.00532	0.00076	0.00111	0.00168	0.00989	0.00083	0.00832
Cluster2											
	Expected short term return	0.24057	0.15152	0.12064	0.32033	0.08501	0.06110	0.27495	0.18855	0.18030	0.39583
	Expected long term return	0.29892	0.29428	0.26969	0.34734	0.23776	0.21098	0.30107	0.29633	0.36700	0.30120
	Risk (standard deviation)	0.47987	0.50224	0.50553	0.64998	0.58731	0.45066	0.43539	0.48685	0.54553	0.45199
	Liquidity	0.00028	0.00104	0.00153	0.00104	0.00155	0.00376	0.00069	0.00084	0.00042	0.00032
Cluster3											
	Expected short term return	0.15810	0.11333	0.10104	0.11453	0.12426	0.08967	0.24725	0.10843	0.26765	0.13041
	Expected long term return	0.19492	0.15927	0.12072	0.14921	0.12386	0.15267	0.23114	0.16003	0.16188	0.15436
	Risk (standard deviation)	0.40147	0.33135	0.24734	0.27544	0.28463	0.25163	0.34483	0.34627	0.28071	0.25564
	Liquidity	0.00548	0.00267	0.00132	0.00413	0.00086	0.00165	0.00473	0.00218	0.00672	0.00972

Table 8.2 Weight calculations of 20 assets for cluster 1 using AHP

			Asset weights									
Criteria	Subcriteria	Weight	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Return												
	SR	0.55714	0.01436	0.00158	0.08248	0.09766	0.10247	0.03299	0.03867	0.01851	0.09124	0.03811
	LR	0.32024	0.04820	0.02898	0.06473	0.06574	0.12128	0.04513	0.05017	0.02965	0.08753	0.07428
	FR	0.12262	0.05829	0.06996	0.08353	0.03398	0.02655	0.03123	0.03573	0.06009	0.02416	0.08318
Risk												
	SD	0.52468	0.04838	0.05489	0.04939	0.04786	0.03220	0.05001	0.04625	0.05359	0.04005	0.04455
	RT	0.33377	0.04720	0.07037	0.08209	0.04933	0.02198	0.03264	0.03702	0.06959	0.02186	0.07084
	MR	0.14156	0.03819	0.08147	0.06234	0.05866	0.01718	0.06185	0.02470	0.08360	0.01509	0.02436
Liquidity	-	-	0.05953	0.00262	0.03500	0.04654	0.13428	0.05550	0.09128	0.03996	0.11199	0.06176
Suitability												
	IS											
	0.63335 IN	0.22230	0.09075	0.08079	0.08220	0.03047	0.01716	0.03057	0.03490	0.05910	0.02240	0.08297
	SO	0.07692	0.08566	0.08056	0.04084	0.04409	0.02005	0.02774	0.02942	0.03113	0.02192	0.08566
	SA	0.55357	0.03980	0.08187	0.06551	0.04343	0.01219	0.04233	0.02075	0.06926	0.01450	0.02135
	SR	0.14721	0.04534	0.02540	0.06181	0.06535	0.14857	0.02406	0.04403	0.02362	0.11641	0.08662
	IO											
	0.26050 AG	0.23595	0.03921	0.09272	0.04156	0.03862	0.01590	0.04232	0.03920	0.02319	0.01594	0.03877
	DE	0.54536	0.08604	0.14502	0.13788	0.03502	0.02003	0.02903	0.03172	0.03275	0.02283	0.04365
	TH	0.10153	0.04611	0.01974	0.02029	0.02482	0.11504	0.04329	0.09031	0.02299	0.11884	0.04748
	R/L	0.11716	0.03947	0.07557	0.04008	0.03868	0.01109	0.03966	0.02063	0.06711	0.01587	0.01751
	IE											
	0.10616 LE	0.62322	0.04987	0.02586	0.02616	0.04472	0.01476	0.04987	0.04943	0.04463	0.02777	0.04914
	EH	0.23949	0.04599	0.02664	0.08116	0.08490	0.15088	0.02464	0.04440	0.02622	0.12132	0.08156
	ED	0.13729	0.05985	0.03081	0.03081	0.10865	0.02143	0.05235	0.10425	0.05912	0.10785	0.05632

			Asset weights									
Criteria	Subcriteria	Weight	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
Return												
	SR	0.55714	0.00881	0.08565	0.07191	0.03070	0.02839	0.07693	0.07430	0.02022	0.01773	0.06729
	LR	0.32024	0.04028	0.02497	0.04321	0.04977	0.03900	0.04958	0.02660	0.04315	0.01832	0.04944
	FR	0.12262	0.04608	0.03958	0.03185	0.04473	0.02864	0.08749	0.04748	0.08749	0.04848	0.03148
Risk												
	SD	0.52468	0.04814	0.06526	0.04748	0.05631	0.06633	0.05383	0.06585	0.03115	0.04383	0.05464
	RT	0.33377	0.04531	0.04225	0.03371	0.05137	0.04200	0.08931	0.04580	0.08081	0.04308	0.02343
	MR	0.14156	0.03976	0.09351	0.03960	0.08707	0.08145	0.07637	0.04889	0.03416	0.01260	0.01914
Liquidity	-	-	0.01800	0.01597	0.05556	0.05181	0.00740	0.01085	0.01635	0.09637	0.00812	0.08110
Suitability												
	IS											
	0.63335 IN	0.22230	0.04575	0.03081	0.02834	0.04268	0.02857	0.08696	0.04694	0.08696	0.04811	0.02357
	SO	0.07692	0.04203	0.02667	0.03048	0.04969	0.03700	0.08162	0.04598	0.14440	0.04335	0.03170
	SA	0.55357	0.03891	0.10000	0.04134	0.06415	0.09697	0.05957	0.09686	0.01254	0.01610	0.06258
	SR	0.14721	0.05023	0.01561	0.02624	0.05126	0.03846	0.04515	0.02447	0.04309	0.01519	0.04909
	IO											
	0.26050 AG	0.23595	0.04254	0.10470	0.04058	0.07270	0.09907	0.07645	0.09164	0.01049	0.01477	0.05962
	DE	0.54536	0.04388	0.02807	0.02974	0.04733	0.03080	0.04683	0.04254	0.08381	0.04134	0.02168
	TH	0.10153	0.01336	0.01387	0.04434	0.04561	0.01236	0.10277	0.09036	0.03409	0.04367	0.05065
	R/L	0.11716	0.03872	0.11279	0.03606	0.05967	0.09989	0.07234	0.11329	0.01125	0.01762	0.07269
	IE											
	0.10616 LE	0.62322	0.10095	0.05108	0.05289	0.14258	0.05290	0.09624	0.05119	0.02590	0.02588	0.01819
	EH	0.23949	0.04560	0.01484	0.02691	0.04429	0.02455	0.04188	0.01694	0.04180	0.01597	0.03951
	ED	0.13729	0.03120	0.05488	0.06038	0.02213	0.05106	0.02447	0.03430	0.03919	0.02128	0.02966

Table 8.3 Weight calculations of 20 assets for cluster 2 using AHP

			Asset weights									
Criteria	Subcriteria	Weight	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Return												
	SR	0.32024	0.04480	0.04154	0.05221	0.02445	0.09027	0.07305	0.03678	0.04469	0.02194	0.04979
	LR	0.55714	0.03597	0.03986	0.04051	0.03698	0.07479	0.05752	0.05197	0.05126	0.03196	0.03355
	FR	0.12262	0.02709	0.03781	0.06781	0.03664	0.07678	0.05489	0.06178	0.06462	0.03103	0.04149
Risk												
	SD	0.54375	0.06580	0.05999	0.05422	0.04718	0.03786	0.03978	0.04076	0.04231	0.05407	0.06767
	RT	0.34595	0.01729	0.02517	0.04316	0.05567	0.07547	0.06169	0.06357	0.06328	0.03775	0.02210
	MR	0.11030	0.01768	0.03606	0.05191	0.05883	0.07454	0.06934	0.06703	0.05471	0.03457	0.02053
Liquidity	-	-	0.00920	0.03657	0.03593	0.03890	0.10227	0.06992	0.18340	0.12050	0.05225	0.01742
Suitability												
	IS											
	0.63335 IN	0.27763	0.01863	0.00825	0.06460	0.03279	0.05938	0.07297	0.06123	0.04195	0.05818	0.00891
	SO	0.06346	0.01858	0.00914	0.08790	0.06855	0.07138	0.08549	0.08092	0.04289	0.04453	0.00822
	SA	0.55526	0.01930	0.00824	0.06191	0.05470	0.05895	0.06069	0.05836	0.04472	0.07558	0.00887
	SR	0.10365	0.01855	0.04598	0.04630	0.02163	0.09002	0.08075	0.05645	0.01062	0.01056	0.05791
	IO											
	0.26050 AG	0.24956	0.01751	0.04188	0.09339	0.04359	0.08589	0.08311	0.06524	0.08284	0.03795	0.00910
	DE	0.54777	0.01742	0.00872	0.07173	0.05017	0.06975	0.08347	0.06262	0.05346	0.06684	0.00835
	TH	0.12761	0.07678	0.04061	0.03972	0.02744	0.08107	0.01969	0.01411	0.01703	0.03613	0.03789
	R/L	0.07506	0.07210	0.06964	0.03573	0.02326	0.01836	0.02518	0.02865	0.02603	0.03463	0.17002
	IE											
	0.10616 LE	0.55714	0.01835	0.03546	0.03842	0.01980	0.08177	0.07404	0.06340	0.01161	0.01637	0.04163
	EH	0.12262	0.01845	0.03905	0.03869	0.02084	0.08465	0.07614	0.06229	0.05731	0.01033	0.01181
	ED	0.32024	0.09969	0.04921	0.05030	0.03764	0.02452	0.02497	0.02937	0.02704	0.04343	0.17371

			Asset weights									
Criteria	Subcriteria	Weight	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
Return												
	SR	0.32024	0.06203	0.03906	0.03110	0.08259	0.02192	0.01575	0.07089	0.04861	0.04649	0.10205
	LR	0.55714	0.05577	0.05490	0.05032	0.06480	0.04436	0.03936	0.05617	0.05529	0.06847	0.05619
	FR	0.12262	0.05938	0.04625	0.03862	0.07142	0.02949	0.02945	0.05687	0.06895	0.06395	0.03567
Risk												
	SD	0.54375	0.05133	0.04904	0.04872	0.03789	0.04194	0.05465	0.05657	0.05059	0.04515	0.05449
	RT	0.34595	0.03786	0.04837	0.04438	0.06359	0.05134	0.05882	0.05892	0.06758	0.06566	0.03833
	MR	0.11030	0.03605	0.05154	0.05274	0.05611	0.05093	0.05437	0.04884	0.06967	0.06292	0.03164
Liquidity	-	-	0.00818	0.03021	0.04454	0.03035	0.04510	0.10917	0.02013	0.02453	0.01225	0.00919
Suitability												
	IS											
	0.63335 IN	0.27763	0.05261	0.04307	0.03897	0.05759	0.06802	0.08862	0.06213	0.08267	0.03520	0.04423
	SO	0.06346	0.06388	0.05584	0.03875	0.03531	0.04083	0.05352	0.05462	0.06550	0.03272	0.04143
	SA	0.55526	0.07282	0.05488	0.03795	0.05027	0.06671	0.04741	0.06688	0.05758	0.04027	0.05392
	SR	0.10365	0.05425	0.03758	0.07251	0.06821	0.04052	0.05280	0.05557	0.05751	0.06141	0.06088
	IO											
	0.26050 AG	0.24956	0.05169	0.05081	0.06249	0.05773	0.03202	0.03420	0.01256	0.04316	0.06261	0.03224
	DE	0.54777	0.06519	0.04560	0.03647	0.04548	0.06943	0.06014	0.04344	0.07114	0.03201	0.03858
	TH	0.12761	0.14825	0.03797	0.03807	0.03586	0.04244	0.07790	0.03727	0.03965	0.07341	0.07869
	R/L	0.07506	0.03696	0.03981	0.03733	0.02396	0.02945	0.03834	0.07007	0.07027	0.07040	0.07980
	IE											
	0.10616 LE	0.55714	0.04435	0.04168	0.06870	0.07019	0.03553	0.06041	0.05682	0.06141	0.07531	0.08472
	EH	0.12262	0.07560	0.07705	0.06184	0.07117	0.03905	0.03245	0.06523	0.04623	0.06392	0.04790
	ED	0.32024	0.05479	0.05204	0.03353	0.02634	0.03555	0.05147	0.05038	0.05454	0.03102	0.05045

Table 8.4 Weight calculations of 20 assets for cluster 3 using AHP

			Asset weights									
Criteria	Subcriteria	Weight	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Return												
	SR	0.45767	0.03511	0.00968	0.04855	0.03383	0.07942	0.09269	0.05159	0.05596	0.04478	0.04369
	LR	0.41601	0.04393	0.05375	0.04808	0.04650	0.05611	0.08691	0.03392	0.03781	0.03316	0.03986
	FR	0.12632	0.03017	0.06344	0.03931	0.03772	0.06506	0.10173	0.01845	0.02292	0.02505	0.03353
Risk												
	SD	0.58889	0.05092	0.03866	0.04710	0.04791	0.05275	0.02702	0.05452	0.05267	0.05798	0.06133
	RT	0.25185	0.04269	0.02388	0.04247	0.04080	0.05156	0.01441	0.08888	0.04233	0.09604	0.09935
	MR	0.15926	0.04178	0.01702	0.09032	0.04657	0.04048	0.02055	0.07835	0.06626	0.06863	0.10661
Liquidity	-	-	0.01294	0.07401	0.02956	0.03694	0.02411	0.18142	0.00867	0.03244	0.00656	0.01359
Suitability												
	IS											
	0.63335 IN	0.22596	0.02969	0.09665	0.09665	0.05531	0.00837	0.09158	0.01544	0.05531	0.09075	0.02969
	SO	0.10462	0.03008	0.05851	0.05851	0.05896	0.00977	0.10572	0.01095	0.06740	0.10581	0.03008
	SA	0.50693	0.02785	0.08182	0.08033	0.05279	0.00857	0.09656	0.00925	0.05378	0.09678	0.02835
	SR	0.16250	0.02733	0.08436	0.10046	0.05201	0.00926	0.08267	0.00896	0.05201	0.09323	0.02733
	IO											
	0.26050 AG	0.23734	0.04819	0.01811	0.02810	0.04637	0.04595	0.00975	0.04494	0.04753	0.09683	0.10348
	DE	0.54536	0.04656	0.01226	0.01652	0.02501	0.04396	0.00874	0.04671	0.04436	0.08611	0.14811
	TH	0.10784	0.02284	0.04280	0.04161	0.04333	0.07148	0.11671	0.01181	0.01169	0.01225	0.01224
	R/L	0.10946	0.03996	0.01853	0.02086	0.02108	0.04066	0.00859	0.07204	0.03856	0.07521	0.12467
	IE											
	0.10616 LE	0.54848	0.02315	0.08239	0.04372	0.04372	0.02874	0.13703	0.01231	0.04511	0.01223	0.02385
	EH	0.21061	0.02625	0.04979	0.09076	0.10464	0.10644	0.08425	0.01253	0.01354	0.01304	0.01319
	ED	0.24091	0.03630	0.01945	0.06864	0.10560	0.01215	0.02052	0.01924	0.06290	0.01888	0.08862

			Asset weights									
Criteria	Subcriteria	Weight	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
Return												
	SR	0.45767	0.05485	0.03932	0.03506	0.03974	0.04311	0.03111	0.08578	0.03762	0.09286	0.04525
	LR	0.41601	0.06303	0.05150	0.03903	0.04825	0.04005	0.04936	0.07474	0.05174	0.05234	0.04991
	FR	0.12632	0.07766	0.05173	0.03047	0.06388	0.03048	0.05023	0.08119	0.06415	0.07185	0.04099
Risk												
	SD	0.58889	0.03737	0.04528	0.06066	0.05447	0.05272	0.05963	0.04351	0.04333	0.05345	0.05869
	RT	0.25185	0.01924	0.02403	0.08609	0.04108	0.03928	0.08683	0.02500	0.02472	0.04780	0.06351
	MR	0.15926	0.01953	0.02644	0.07075	0.04218	0.04403	0.07441	0.02301	0.04139	0.04024	0.04146
Liquidity	-	-	0.08053	0.03922	0.01940	0.06063	0.01260	0.02431	0.06946	0.03200	0.09877	0.14283
Suitability												
	IS											
	0.63335 IN	0.22596	0.05531	0.09158	0.01306	0.02969	0.01069	0.01502	0.09158	0.05531	0.01302	0.05531
	SO	0.10462	0.10497	0.05953	0.01513	0.02981	0.00786	0.01565	0.09627	0.06036	0.01425	0.06036
	SA	0.50693	0.09579	0.07688	0.01101	0.02782	0.00884	0.01463	0.10417	0.05895	0.00876	0.05708
	SR	0.16250	0.08267	0.09278	0.00892	0.02733	0.00960	0.01424	0.09156	0.07454	0.00872	0.05201
	IO											
	0.26050 AG	0.23734	0.01564	0.02486	0.08290	0.04547	0.04865	0.02836	0.02536	0.04725	0.09254	0.09973
	DE	0.54536	0.01255	0.01749	0.08526	0.08355	0.04521	0.01776	0.01778	0.04497	0.08355	0.11354
	TH	0.10784	0.09509	0.04819	0.02343	0.02308	0.04176	0.07304	0.08806	0.07483	0.07052	0.07524
	R/L	0.10946	0.01247	0.02065	0.07591	0.07591	0.04038	0.07521	0.02065	0.02272	0.07788	0.11807
	IE											
	0.10616 LE	0.54848	0.01213	0.04453	0.04449	0.07867	0.02385	0.04511	0.07797	0.04511	0.08025	0.09566
	EH	0.21061	0.02747	0.04990	0.08095	0.02595	0.02773	0.04956	0.09481	0.02692	0.05179	0.05049
	ED	0.24091	0.09013	0.05461	0.09669	0.06392	0.09142	0.06388	0.01672	0.01783	0.03281	0.01968

Table 8.5 AHP-weights for criteria and assets for cluster 1, cluster 2 & cluster 3

Cluster	Criteria	Local weights (performance scores) of the assets										Criteria weight
		A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	(relative importance)
Cluster1												
	Return	0.03058	0.01874	0.07692	0.07963	0.09918	0.03666	0.04199	0.02717	0.08183	0.05522	0.19281
	Risk	0.04655	0.06382	0.06214	0.04988	0.02667	0.04589	0.04012	0.06318	0.03045	0.05047	0.13521
	Liquidity	0.05953	0.00262	0.03500	0.04654	0.13428	0.05550	0.09128	0.03996	0.11199	0.06176	0.57190
	Suitability	0.05753	0.07834	0.07044	0.04364	0.03385	0.03651	0.03358	0.04954	0.03450	0.04793	0.10008
Cluster2												
	Return	0.03771	0.04015	0.04760	0.03292	0.07999	0.06217	0.04831	0.05080	0.02864	0.03973	0.47625
	Risk	0.04371	0.04531	0.05014	0.05140	0.05491	0.05062	0.05155	0.05093	0.04627	0.04670	0.35620
	Liquidity	0.00920	0.03657	0.03593	0.03890	0.10227	0.06992	0.18340	0.12050	0.05225	0.01742	0.07040
	Suitability	0.02433	0.01869	0.06251	0.04326	0.06531	0.06759	0.05801	0.04196	0.05576	0.02370	0.09715
Cluster3												
	Return	0.03816	0.03480	0.04719	0.03960	0.06791	0.09143	0.04006	0.04424	0.03745	0.04081	0.35557
	Risk	0.04739	0.03149	0.05281	0.04591	0.05050	0.02281	0.06697	0.05223	0.06927	0.07812	0.48111
	Liquidity	0.01294	0.07401	0.02956	0.03694	0.02411	0.18142	0.00867	0.03244	0.00656	0.01359	0.09056
	Suitability	0.03223	0.06366	0.06602	0.04995	0.02217	0.07537	0.02012	0.05021	0.08286	0.05344	0.07276

Cluster	Critera	Local weights (performance scores) of the assets										Criteria weight
		A11	A12	A13	A14	A15	A16	A17	A18	A19	A20	(relative importance)
Cluster1												
	Return	0.02346	0.06057	0.05781	0.03853	0.03182	0.06946	0.05573	0.03581	0.02169	0.05718	0.19281
	Risk	0.04601	0.06158	0.04177	0.05902	0.06035	0.06886	0.05675	0.04815	0.03916	0.03920	0.13521
	Liquidity	0.01800	0.01597	0.05556	0.05181	0.00740	0.01085	0.01635	0.09637	0.00812	0.08110	0.57190
	Suitability	0.04549	0.06094	0.03647	0.06080	0.06212	0.06538	0.06693	0.04484	0.02685	0.04432	0.10008
Cluster2												
	Return	0.05822	0.04877	0.04273	0.07131	0.03535	0.03058	0.06097	0.05482	0.06088	0.06836	0.47625
	Risk	0.04498	0.04908	0.04766	0.04879	0.04618	0.05606	0.05653	0.05857	0.05421	0.04638	0.35620
	Liquidity	0.00818	0.03021	0.04454	0.03035	0.04510	0.10917	0.02013	0.02453	0.01225	0.00919	0.07040
	Suitability	0.06477	0.04867	0.04379	0.05158	0.05751	0.05776	0.05584	0.06293	0.04449	0.05153	0.09715
Cluster3												
	Return	0.06113	0.04595	0.03613	0.04633	0.04024	0.04112	0.08061	0.04685	0.07335	0.04665	0.35557
	Risk	0.02996	0.03693	0.06867	0.04914	0.04795	0.06883	0.03558	0.03834	0.04992	0.05716	0.48111
	Liquidity	0.08053	0.03922	0.01940	0.06063	0.01260	0.02431	0.06946	0.03200	0.09877	0.14283	0.09056
	Suitability	0.06354	0.06235	0.03426	0.04225	0.02198	0.02319	0.07660	0.05426	0.03498	0.07057	0.07276

8.4.2 Asset Allocation

The 20 financial assets of each cluster comprise the population from which we attempt to construct a portfolio containing 8 assets with the corresponding upper and lower bounds of capital budget allocation. We present computational results corresponding to three types of investor behaviour.

- **Cluster 1 for liquidity seekers**

Step 1: To find the optimal asset allocation, we first formulate the model P(8.1) using the the local weights (performance scores) of the assets with respect to the four key asset allocation criteria listed in Table 8.5, $h = 8$, $l_i = 0.01$, and $u_i = 0.6$, $i = 1, 2, \ldots, 20$.

$$\begin{aligned}
\max\ f_1(x) = {} & 0.03058x_1 + 0.01874x_2 + 0.07692x_3 + 0.07963x_4 + 0.09918x_5 \\
& +0.03666x_6 + 0.04199x_7 + 0.02717x_8 + 0.08183x_9 + 0.05522x_{10} + 0.02346x_{11} \\
& +0.06057x_{12} + 0.05781x_{13} + 0.03853x_{14} + 0.03182x_{15} + 0.06946x_{16} \\
& +0.05573x_{17} + 0.03581x_{18} + 0.02169x_{19} + 0.05718x_{20} \\
\max\ f_2(x) = {} & 0.04655x_1 + 0.06382x_2 + 0.06214x_3 + 0.04988x_4 + 0.02667x_5 \\
& +0.04589x_6 + 0.04012x_7 + 0.06318x_8 + 0.03045x_9 + 0.05047x_{10} + 0.04601x_{11} \\
& +0.06158x_{12} + 0.04177x_{13} + 0.05902x_{14} + 0.06035x_{15} + 0.06886x_{16} \\
& +0.05675x_{17} + 0.04815x_{18} + 0.03916x_{19} + 0.03920x_{20} \\
\max\ f_3(x) = {} & 0.05953x_1 + 0.00262x_2 + 0.035x_3 + 0.04654x_4 + 0.13428x_5 \\
& +0.0555x_6 + 0.09128x_7 + 0.03996x_8 + 0.11199x_9 + 0.06176x_{10} + 0.018x_{11} \\
& +0.01597x_{12} + 0.05556x_{13} + 0.05181x_{14} + 0.0074x_{15} + 0.01085x_{16} \\
& +0.01635x_{17} + 0.09637x_{18} + 0.00812x_{19} + 0.08110x_{20} \\
\max\ f_4(x) = {} & 0.05753x_1 + 0.07834x_2 + 0.07044x_3 + 0.04364x_4 + 0.03385x_5 \\
& +0.03651x_6 + 0.03358x_7 + 0.04954x_8 + 0.0345x_9 + 0.04793x_{10} + 0.04549x_{11} \\
& +0.06094x_{12} + 0.03647x_{13} + 0.06080x_{14} + 0.06212x_{15} + 0.06538x_{16} \\
& +0.06693x_{17} + 0.04484x_{18} + 0.02685x_{19} + 0.04432x_{20}
\end{aligned}$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} + x_{14} + x_{15}$$
$$+x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,, \tag{8.7}$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} + y_{14} + y_{15}$$
$$+y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 8\,, \tag{8.8}$$
$$x_i - 0.01y_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,, \tag{8.9}$$
$$x_i - 0.6y_i \leq 0\,, \quad i = 1, 2, \ldots, 20\,, \tag{8.10}$$
$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 20\,, \tag{8.11}$$
$$x_i \geq 0\,, \quad i = 1, 2, \ldots, 20\,. \tag{8.12}$$

The above model is solved using the fuzzy interactive approach developed in Section 8.3. The models are coded and solved using LINGO 12.0.

Step 2: We determine the worst lower bound and best upper bound for return, risk, liquidity and suitability criteria, respectively, by solving the following single-objective problems.

For Return Criterion

$$\begin{aligned}\max\ f_1(x) &= 0.03058x_1 + 0.01874x_2 + 0.07692x_3 + 0.07963x_4 + 0.09918x_5\\ &+0.03666x_6 + 0.04199x_7 + 0.02717x_8 + 0.08183x_9 + 0.05522x_{10} + 0.02346x_{11}\\ &+0.06057x_{12} + 0.05781x_{13} + 0.03853x_{14} + 0.03182x_{15} + 0.06946x_{16}\\ &+0.05573x_{17} + 0.03581x_{18} + 0.02169x_{19} + 0.05718x_{20}\end{aligned}$$

subject to

Constraints (8.7)-(8.12) .

The obtained solution is denoted as $x^1 = (x_1, x_2, \ldots, x_{20})$ and is provided in Table 8.6.

For Risk Criterion

$$\begin{aligned}\max\ f_2(x) &= 0.04655x_1 + 0.06382x_2 + 0.06214x_3 + 0.04988x_4 + 0.02667x_5\\ &+0.04589x_6 + 0.04012x_7 + 0.06318x_8 + 0.03045x_9 + 0.05047x_{10} + 0.04601x_{11}\\ &+0.06158x_{12} + 0.04177x_{13} + 0.05902x_{14} + 0.06035x_{15} + 0.06886x_{16}\\ &+0.05675x_{17} + 0.04815x_{18} + 0.03916x_{19} + 0.03920x_{20}\end{aligned}$$

subject to

Constraints (8.7)-(8.12) .

The obtained solution is denoted as $x^2 = (x_1, x_2, \ldots, x_{20})$ and is provided in Table 8.6.

For Liquidity Criterion

$$\begin{aligned}\max\ f_3(x) &= 0.05953x_1 + 0.00262x_2 + 0.035x_3 + 0.04654x_4 + 0.13428x_5\\ &+0.0555x_6 + 0.09128x_7 + 0.03996x_8 + 0.11199x_9 + 0.06176x_{10} + 0.018x_{11}\\ &+0.01597x_{12} + 0.05556x_{13} + 0.05181x_{14} + 0.0074x_{15} + 0.01085x_{16}\\ &+0.01635x_{17} + 0.09637x_{18} + 0.00812x_{19} + 0.08110x_{20}\end{aligned}$$

subject to

Constraints (8.7)-(8.12) .

The obtained solution is denoted as $x^3 = (x_1, x_2, \ldots, x_{20})$ and is provided in Table 8.6.

For Suitability Criterion

$$\begin{aligned}\max\ f_4(x) = {} & 0.05753x_1 + 0.07834x_2 + 0.07044x_3 + 0.04364x_4 + 0.03385x_5 \\ & +0.03651x_6 + 0.03358x_7 + 0.04954x_8 + 0.0345x_9 + 0.04793x_{10} + 0.04549x_{11} \\ & +0.06094x_{12} + 0.03647x_{13} + 0.06080x_{14} + 0.06212x_{15} + 0.06538x_{16} \\ & +0.06693x_{17} + 0.04484x_{18} + 0.02685x_{19} + 0.04432x_{20}\end{aligned}$$

subject to

Constraints (8.7)-(8.12).

The obtained solution is denoted as $x^4 = (x_1, x_2, \ldots, x_{20})$ and is provided in Table 8.6.

Table 8.6 The proportions of the assets in the obtained portfolios corresponding to single-objective problems for liquidity seekers

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
x^1	0.0	0.0	0.02	0.253	0.35	0.0	0.0	0.0	0.3	0.0
x^2	0.0	0.263	0.020	0.0	0.0	0.0	0.0	0.028	0.0	0.026
x^3	0.0	0.0	0.0	0.0	0.35	0.0	0.02	0.0	0.3	0.026
x^4	0.0	0.4	0.35	0.0	0.0	0.0	0.0	0.028	0.0	0.0
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0.0	0.0	0.023	0.0	0.0	0.016	0.023	0.0	0.0	0.015
	0.0	0.03	0.0	0.0	0.0	0.6	0.023	0.0	0.01	0.0
	0.0	0.0	0.023	0.0	0.0	0.0	0.0	0.256	0.01	0.015
	0.0	0.03	0.0	0.0	0.0	0.016	0.151	0.0	0.01	0.015

Step 3: All the objective functions are evaluated at the obtained solutions, i.e., x^1, x^2, x^3 and x^4. The objective function values are provided in Table 8.7.

Table 8.7 Objective function values corresponding to return, risk, liquidity and suitability criteria for liquidity seekers

	x^1	x^2	x^3	x^4
Return($f_1(x)$)	0.08553^a	0.05366	0.07311	0.04760^b
Risk($f_2(x)$)	0.03629	0.06597^a	0.03485^b	0.06154
Liquidity($f_3(x)$)	0.09612	0.01156^b	0.11127^a	0.01883
Suitability($f_4(x)$)	0.03874	0.06751	0.03736^b	0.07129^a

^{a}Upper bound
^{b}Lower bound

Now, we define the worst lower bound and best upper bound of each criterion as follows:

$$f_1^L(=0.04760) \le f_1(x) \le f_1^R(=0.08553)\,,$$
$$f_2^L(=0.03485) \le f_2(x) \le f_2^R(=0.06597)\,,$$
$$f_3^L(=0.01156) \le f_3(x) \le f_3^R(=0.11127)\,,$$
$$f_4^L(=0.03736) \le f_4(x) \le f_4^R(=0.06751)\,.$$

Step 4: The linear membership function of the goal of return is

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \ge 0.08553\,, \\ \dfrac{f_1(x) - 0.04760}{0.03793}, & \text{if } 0.04760 \le f_1(x) \le 0.08553\,, \\ 0, & \text{if } f_1(x) \le 0.04760\,. \end{cases}$$

The linear membership function of the goal of risk is

$$\mu_{f_2}(x) = \begin{cases} 1, & \text{if } f_2(x) \ge 0.06597\,, \\ \dfrac{f_2(x) - 0.03485}{0.03112}, & \text{if } 0.03485 \le f_2(x) \le 0.06597\,, \\ 0, & \text{if } f_2(x) \le 0.03485\,. \end{cases}$$

The linear membership function of the goal of liquidity is

$$\mu_{f_3}(x) = \begin{cases} 1, & \text{if } f_3(x) \ge 0.11127\,, \\ \dfrac{f_3(x) - 0.01156}{0.09971}, & \text{if } 0.01156 \le f_3(x) \le 0.11127\,, \\ 0, & \text{if } f_3(x) \le 0.01156\,. \end{cases}$$

The linear membership function of the goal of suitability is

$$\mu_{f_4}(x) = \begin{cases} 1, & \text{if } f_4(x) \ge 0.06751\,, \\ \dfrac{f_4(x) - 0.03736}{0.03015}, & \text{if } 0.03736 \le f_4(x) \le 0.06751\,, \\ 0, & \text{if } f_4(x) \le 0.03736\,. \end{cases}$$

Steps 5 & 6: Using the weights of the four key criteria calculated using AHP from Table 8.5, under the heading 'Criteria weight', we convert the above multiobjective problem into a single-objective problem as

$$
\begin{aligned}
&\max\ 0.19281\alpha_1 + 0.13521\alpha_2 + 0.57190\alpha_3 + 0.10008\alpha_4\\
&\text{subject to}\\
&0.03058x_1 + 0.01874x_2 + 0.07692x_3 + 0.07963x_4 + 0.09918x_5\\
&\quad +0.03666x_6 + 0.04199x_7 + 0.02717x_8 + 0.08183x_9 + 0.05522x_{10} + 0.02346x_{11}\\
&\quad +0.06057x_{12} + 0.05781x_{13} + 0.03853x_{14} + 0.03182x_{15} + 0.06946x_{16}\\
&\quad +0.05573x_{17} + 0.03581x_{18} + 0.02169x_{19} + 0.05718x_{20} - 0.03793\alpha_1 \geq 0.04760\,,\\
&0.04655x_1 + 0.06382x_2 + 0.06214x_3 + 0.04988x_4 + 0.02667x_5\\
&\quad +0.04589x_6 + 0.04012x_7 + 0.06318x_8 + 0.03045x_9 + 0.05047x_{10} + 0.04601x_{11}\\
&\quad +0.06158x_{12} + 0.04177x_{13} + 0.05902x_{14} + 0.06035x_{15} + 0.06886x_{16}\\
&\quad +0.05675x_{17} + 0.04815x_{18} + 0.03916x_{19} + 0.03920x_{20} - 0.03112\alpha_2 \geq 0.03485\,,\\
&0.05953x_1 + 0.00262x_2 + 0.035x_3 + 0.04654x_4 + 0.13428x_5\\
&\quad +0.0555x_6 + 0.09128x_7 + 0.03996x_8 + 0.11199x_9 + 0.06176x_{10} + 0.018x_{11}\\
&\quad +0.01597x_{12} + 0.05556x_{13} + 0.05181x_{14} + 0.0074x_{15} + 0.01085x_{16}\\
&\quad +0.01635x_{17} + 0.09637x_{18} + 0.00812x_{19} + 0.08110x_{20} - 0.09971\alpha_3 \geq 0.01156\,,\\
&0.05753x_1 + 0.07834x_2 + 0.07044x_3 + 0.04364x_4 + 0.03385x_5\\
&\quad +0.03651x_6 + 0.03358x_7 + 0.04954x_8 + 0.0345x_9 + 0.04793x_{10} + 0.04549x_{11}\\
&\quad +0.06094x_{12} + 0.03647x_{13} + 0.06080x_{14} + 0.06212x_{15} + 0.06538x_{16}\\
&\quad +0.06693x_{17} + 0.04484x_{18} + 0.02685x_{19} + 0.04432x_{20} - 0.03015\alpha_4 \geq 0.03736\,,\\
&0 \leq \alpha_1 \leq 1\,,\\
&0 \leq \alpha_2 \leq 1\,,\\
&0 \leq \alpha_3 \leq 1\,,\\
&0 \leq \alpha_4 \leq 1\,,\\
&\text{and Constraints (8.7)-(8.12)}\,.
\end{aligned}
$$

The corresponding computational results are summarized in Table 8.8. Table 8.9 present proportions of the assets in the obtained portfolios. The achievement levels of the various objectives are consistent with the investor preferences.

Table 8.8 Results of the model P(8.2) for liquidity seekers

	Objective function value	Membership function value
Return($f_1(x)$)	0.06655	0.49961
Risk($f_2(x)$)	0.04508	0.32862
Liquidity($f_3(x)$)	0.08539	0.74044
Suitability($f_4(x)$)	0.04773	0.30566

Table 8.9 The proportions of the assets in the obtained portfolio for liquidity seekers

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Portfolio	0	0	0.264	0.025	0.10	0	0.02	0	0.25	0.026
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0	0	0	0	0	0	0	0.30	0	0.015

Table 8.10 Attainment values of the quantitative criteria for liquidity seekers

Criteria	Value
Short term return (SR)	0.13794
Long term return (LR)	0.16136
Risk (RI)	0.56233
Liquidity (LI)	0.00876

The values of the quantitative measures of asset performance corresponding to the obtained solution are given in Table 8.10.

Note that it is possible to further improve portfolio performance on individual objective(s) as per investor preferences. However, it must be understood that because of the multiobjective nature of the problem there may be a compensatory variation on the other performance measures. After getting solution, suppose the investor is not satisfied with the first objective, i.e., f_1 (return criterion). We compare the present lower bound of the return objective with the obtained new value achieved for f_1. Since the new value for f_1 listed in Table 8.8 is higher than the present lower bound, i.e., $f_1(x)(= 0.06655) > f_1^L(= 0.04760)$, the lower bound is revised. Revise the linear membership function for return criterion as

$$\mu_{f_1}(x) = \begin{cases} 1, & \text{if } f_1(x) \geq 0.08553\,, \\ \dfrac{f_1(x) - 0.06655}{0.01898}, & \text{if } 0.06655 \leq f_1(x) \leq 0.08553\,, \\ 0, & \text{if } f_1(x) \leq 0.06655\,. \end{cases}$$

Now, the above mentioned single-objective problem is resolved with the new membership function in respect of return and keeping the other parameters as is. The procedure is continued until the investor is satisfied with the obtained portfolio. The solutions of all the iterations performed are given in Table 8.11. Table 8.11 also shows the revised lower bounds of the various objective functions at each iteration. Table 8.12 present attainment values of the quantitative measures of asset performance and the proportions of the assets in the obtained portfolios.

Table 8.11 Iterative results for liquidity seekers

	Iteration number				
	1	2	3	4	5
$f_1(x)$	0.07930 $(0.06655 \leq f_1(x) \leq 0.08553)$	0.07009 $(0.04760 \leq f_1(x) \leq 0.08553)$	0.05745 $(0.04760 \leq f_1(x) \leq 0.08553)$	0.07050 $(0.04760 \leq f_1(x) \leq 0.08553)$	0.06627 $(0.04760 \leq f_1(x) \leq 0.08553)$
$f_2(x)$	0.04695 $(0.03485 \leq f_2(x) \leq 0.06597)$	0.04628 $(0.03485 \leq f_2(x) \leq 0.06597)$	0.03950 $(0.03485 \leq f_2(x) \leq 0.06597)$	0.04677 $(0.04508 \leq f_2(x) \leq 0.06597)$	0.04662 $(0.04508 \leq f_2(x) \leq 0.06597)$
$f_3(x)$	0.06888 $(0.01156 \leq f_3(x) \leq 0.11127)$	0.08011 $(0.01156 \leq f_3(x) \leq 0.11127)$	0.09942 $(0.08539 \leq f_3(x) \leq 0.11127)$	0.07884 $(0.01156 \leq f_3(x) \leq 0.11127)$	0.08163 $(0.01156 \leq f_3(x) \leq 0.11127)$
$f_4(x)$	0.05003 $(0.03736 \leq f_4(x) \leq 0.07129)$	0.04993 $(0.04773 \leq f_4(x) \leq 0.07129)$	0.03855 $(0.03736 \leq f_4(x) \leq 0.07129)$	0.05049 $(0.04773 \leq f_4(x) \leq 0.07129)$	0.04948 $(0.03736 \leq f_4(x) \leq 0.07129)$

Table 8.12 The proportions of the assets in the obtained portfolios corresponding to investor preferences (liquidity seekers)

Iterations	SR	LR	RI	LI	Allocation									
					A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Iteration 1	0.18299	0.17632	0.48797	0.00707	0	0	0.35	0.223	0.10	0	0.02	0	0.25	0.026
Iteration 2	0.14911	0.16568	0.54038	0.00822	0	0	0.35	0.025	0.10	0	0.02	0	0.25	0.026
Iteration 3	0.11415	0.15182	0.56964	0.01020	0	0	0.02	0	0.10	0	0.266	0	0.25	0.026
Iteration 4	0.15023	0.16559	0.54024	0.00809	0	0	0.35	0.025	0.10	0	0	0	0.25	0.026
Iteration 5	0.13696	0.15864	0.55717	0.00838	0	0	0.312	0.025	0.10	0	0.02	0	0.20	0.026
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0	0	0	0	0	0.016	0	0	0	0.015
					0	0	0	0	0	0	0	0.214	0	0.015
					0	0	0.023	0	0	0	0	0.30	0	0.015
					0	0	0	0	0	0	0	0.218	0	0.015
					0	0	0	0	0	0	0	0.30	0	0.017

• Cluster 2 for return seekers

The data set for the values of the best upper bound and the worst lower bound are given in the pay-off table (Table 8.13).

Table 8.13 Objective function values corresponding to return, risk, liquidity and suitability criteria for return seekers

	x^1	x^2	x^3	x^4
Return($f_1(x)$)	0.07270[a]	0.05655	0.04704[b]	0.06661
Risk($f_2(x)$)	0.05104[b]	0.05713[a]	0.05207	0.05158
Liquidity($f_3(x)$)	0.05759	0.02928[b]	0.13116[a]	0.07202
Suitability($f_4(x)$)	0.05747	0.05934	0.05123[b]	0.06554[a]

[a]Upper bound
[b]Lower bound

The model P(8.2) is formulated using the weights of the key asset allocation criteria listed in Table 8.5. We obtain portfolio selection by solving model P(8.2) as discussed above. The corresponding computational results are summarized in Table 8.14. Table 8.15 present proportions of the assets in the obtained portfolios. The achievement levels of the various objectives are consistent with the investor preferences.

Table 8.14 Results of the model P(8.2) for return seekers

	Objective function value	Membership function value
Return($f_1(x)$)	0.06313	0.62692
Risk($f_2(x)$)	0.05479	0.61616
Liquidity($f_3(x)$)	0.04993	0.20272
Suitability($f_4(x)$)	0.05925	0.56042

The values of the quantitative measures of asset performance corresponding to the obtained solution are given in Table 8.16.

The results of the various iterations performed in order to further improve portfolio performance on individual objective(s) as per investor preferences are given in Table 8.17 which also shows the revised lower bounds of the various objective functions at each iteration. Table 8.18 present attainment values of the quantitative measures of asset performance and the proportions of the assets in the obtained portfolios corresponding to the iterations performed.

Table 8.15 The proportions of the assets in the obtained portfolio for return seekers

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Portfolio	0	0	0	0	0.20	0.19	0.02	0	0	0
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0	0	0	0	0	0.016	0.17	0.20	0.189	0.015

Table 8.16 Attainment values of the quantitative criteria for return seekers

Criteria	Value
Short term return (SR)	0.25215
Long term return (LR)	0.33203
Risk (RI)	0.54833
Liquidity (LI)	0.00172

- **Cluster 3 for safety seekers**

As demonstrated above, the data set for the values of the best upper bound and the worst lower bound are given in the pay-off table (Table 8.19).

Table 8.17 Iterative results for return seekers

	Iteration number		
	1	2	3
$f_1(x)$	0.06946 $(0.06313 \le f_1(x) \le 0.07270)$	0.06254 $(0.04704 \le f_1(x) \le 0.07270)$	0.06406 $(0.04704 \le f_1(x) \le 0.07270)$
$f_2(x)$	0.05204 $(0.05104 \le f_2(x) \le 0.05713)$	0.05542 $(0.05479 \le f_2(x) \le 0.05713)$	0.05437 $(0.05104 \le f_2(x) \le 0.05713)$
$f_3(x)$	0.04620 $(0.02928 \le f_3(x) \le 0.13116)$	0.04094 $(0.02928 \le f_3(x) \le 0.13116)$	0.05179 $(0.02928 \le f_3(x) \le 0.13116)$
$f_4(x)$	0.05619 $(0.05123 \le f_4(x) \le 0.06554)$	0.05569 $(0.05123 \le f_4(x) \le 0.06554)$	0.05985 $(0.05925 \le f_4(x) \le 0.06554)$
	Iteration number		
	4	5	6
$f_1(x)$	0.06504 $(0.04704 \le f_1(x) \le 0.07270)$	0.06313 $(0.06313 \le f_1(x) \le 0.07270)$	0.05441 $(0.04704 \le f_1(x) \le 0.07270)$
$f_2(x)$	0.05386 $(0.05104 \le f_2(x) \le 0.05713)$	0.05527 $(0.05479 \le f_2(x) \le 0.05713)$	0.05592 $(0.05479 \le f_2(x) \le 0.05713)$
$f_3(x)$	0.05349 $(0.04993 \le f_3(x) \le 0.13116)$	0.03934 $(0.02928 \le f_3(x) \le 0.13116)$	0.06692 $(0.02928 \le f_3(x) \le 0.13116)$
$f_4(x)$	0.06052 $(0.05925 \le f_4(x) \le 0.06554)$	0.05559 $(0.05123 \le f_4(x) \le 0.06554)$	0.05925 $(0.05925 \le f_4(x) \le 0.06554)$

Table 8.18 The proportions of the assets in the obtained portfolios corresponding to investor preferences (return seekers)

Iterations	ST	LT	RI	LI	Allocation									
					A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Iteration 1	0.30591	0.34427	0.59813	0.00159	0	0	0	0	0.20	0.025	0.02	0	0	0
Iteration 2	0.23236	0.33970	0.53575	0.00141	0	0	0.02	0	0.20	0.025	0.02	0	0	0
Iteration 3	0.26292	0.33105	0.56018	0.00178	0	0	0	0	0.20	0	0.02	0	0	0
Iteration 4	0.27608	0.32920	0.57	0.00184	0	0	0	0	0.20	0.19	0.02	0	0	0
Iteration 5	0.23750	0.34121	0.53587	0.00135	0	0	0.02	0	0.20	0.025	0.02	0	0	0
Iteration 6	0.20040	0.29787	0.51032	0.00230	0	0	0.02	0	0.20	0.025	0.02	0	0	0
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0	0	0	0.50	0	0	0.17	0.06	0.01	0.015
					0	0	0	0	0	0.016	0.17	0.20	0.349	0
					0	0	0	0.10	0	0.016	0.17	0.20	0.104	0
					0	0	0	0.194	0	0.016	0.17	0.20	0.01	0
					0	0	0	0	0	0	0.17	0.20	0.35	0.015
					0	0	0	0	0	0.28411	0.17	0.20	0.08089	0

Table 8.19 Objective function values corresponding to return, risk, liquidity and suitability criteria for safety seekers

	x^1	x^2	x^3	x^4
Return($f_1(x)$)	0.08203[a]	0.04030[b]	0.07111	0.06486
Risk($f_2(x)$)	0.03188[b]	0.07149[a]	0.04049	0.04600
Liquidity($f_3(x)$)	0.11370	0.01744[b]	0.13965[a]	0.05352
Suitability($f_4(x)$)	0.07298	0.05356[b]	0.06410	0.07678[a]

[a]Upper bound
[b]Lower bound

Using the weights of the key asset allocation criteria listed in Table 8.5, we obtain portfolio selection by solving model P(8.2) as discussed above. The corresponding computational results are summarized in Table 8.20. Table 8.21 present proportions of the assets in the obtained portfolio. The achievement levels of the various objectives are consistent with the investor preferences.

Table 8.20 Results of the model P(8.2) for safety seekers

	Objective function value	Membership function value
Return($f_1(x)$)	0.06209	0.52207
Risk($f_2(x)$)	0.05368	0.55048
Liquidity($f_3(x)$)	0.07432	0.46541
Suitability($f_4(x)$)	0.06230	0.37651

Table 8.21 The proportions of the assets in the obtained portfolio for safety seekers

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Portfolio	0	0	0	0	0.019	0.25	0	0	0.156	0.30
	A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
	0	0	0	0	0	0.02	0.10	0	0.14	0.015

Table 8.22 Attainment values of the quantitative criteria for safety seekers

Criteria	Value
Short term return (SR)	0.19499
Long term return (LR)	0.17462
Risk (RI)	0.34066
Liquidity (LI)	0.00506

The values of the quantitative measures of the asset performance corresponding to the obtained solution are given in Table 8.22.

The results of the various iterations performed in order to further improve portfolio performance on individual objective(s) as per investor preferences are given in Table 8.23. Table 8.23 also shows the revised lower bounds of the various objective functions at each iteration. Table 8.24 present attainment values of the quantitative measures of asset performance and the proportions of the assets in the obtained portfolios corresponding to the iterations performed.

A comparison of the solutions listed in Tables 8.10, 8.16 and 8.22 highlights that if investors are liquidity seekers they will obtain a higher level of liquidity in comparison to return seekers and safety seekers, albeit they may have to settle for similar variability of return/risk. If the investors are return seekers they will obtain a higher level of expected returns in comparison to liquidity seekers and safety seekers, but that supposes assuming a higher risk level. If the investors are safety seekers, they will obtain a lower level of risk in comparison to liquidity seekers and return seekers, but that supposes accepting medium level of expected returns.

The computational results indicate that the model presented here is capable of yielding optimal portfolios not only for each category of investors but can also accommodate individual preferences within each category following an interactive procedure. It may be noted that by revising lower bounds of the linear membership functions, we can obtain different portfolio constructions by solving the model P(8.2). Further, it is important to point out that for some choices of the membership function there may be no improvement in solution with the revised lower bound(s). In such instances, we will have to modify the lower bound(s) for the various scenarios to find a satisfactory solution.

Table 8.23 Iterative results for safety seekers

	Iteration number		
	1	2	3
$f_1(x)$	0.07905	0.04165	0.07856
	$(0.06209 \le f_1(x) \le 0.08203)$	$(0.04030 \le f_1(x) \le 0.08203)$	$(0.06209 \le f_1(x) \le 0.08203)$
$f_2(x)$	0.03712	0.06909	0.03629
	$(0.03188 \le f_2(x) \le 0.07149)$	$(0.05368 \le f_2(x) \le 0.07149)$	$(0.03188 \le f_2(x) \le 0.07149)$
$f_3(x)$	0.09864	0.03639	0.09234
	$(0.01744 \le f_3(x) \le 0.13965)$	$(0.01744 \le f_3(x) \le 0.13965)$	$(0.01744 \le f_3(x) \le 0.13965)$
$f_4(x)$	0.06746	0.05356	0.07351
	$(0.05356 \le f_4(x) \le 0.07678)$	$(0.05356 \le f_4(x) \le 0.07678)$	$(0.06230 \le f_4(x) \le 0.07678)$

	Iteration number		
	4	5	6
$f_1(x)$	0.07228	0.07926	0.05209
	$(0.04030 \le f_1(x) \le 0.08203)$	$(0.06209 \le f_1(x) \le 0.08203)$	$(0.04030 \le f_1(x) \le 0.08203)$
$f_2(x)$	0.04028	0.03574	0.05921
	$(0.03188 \le f_2(x) \le 0.07149)$	$(0.03188 \le f_2(x) \le 0.07149)$	$(0.05368 \le f_2(x) \le 0.07149)$
$f_3(x)$	0.10591	0.09332	0.07432
	$(0.07432 \le f_3(x) \le 0.13965)$	$(0.07432 \le f_3(x) \le 0.13965)$	$(0.07432 \le f_3(x) \le 0.13965)$
$f_4(x)$	0.07239	0.07216	0.06608
	$(0.06230 \le f_4(x) \le 0.07678)$	$(0.06230 \le f_4(x) \le 0.07678)$	$(0.06230 \le f_4(x) \le 0.07678)$

Table 8.24 The proportions of the assets in the obtained portfolios corresponding to investor preferences (safety seekers)

Iterations	ST	LT	RI	LI	Allocation									
					A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Iteration 1	0.24453	0.22259	0.38101	0.00671	0	0	0.02	0	0.019	0.25	0	0	0	0.026
Iteration 2	0.12056	0.13223	0.25403	0.00248	0	0	0.02	0	0.019	0	0.02	0	0.21	0.30
Iteration 3	0.23753	0.22779	0.38697	0.00628	0	0	0.02	0	0.019	0.25	0	0	0.035	0.026
Iteration 4	0.21592	0.21358	0.37047	0.00721	0	0	0.02	0	0.019	0.25	0	0	0.035	0.026
Iteration 5	0.23970	0.22976	0.38859	0.00635	0	0	0.02	0	0.019	0.25	0.02	0	0	0.026
Iteration 6	0.15698	0.15693	0.31820	0.00506	0	0	0.02	0	0	0.213	0.02	0	0.204	0.30
					A11	A12	A13	A14	A15	A16	A17	A18	A19	A20
					0	0	0	0	0	0.02	0.51	0	0.14	0.015
					0	0	0.023	0	0	0.24477	0	0	0	0.16323
					0	0	0	0	0	0.02	0.615	0	0	0.015
					0	0	0	0	0	0.02	0.43	0	0	0.20
					0	0	0	0	0	0.02	0.63	0	0	0.015
					0	0	0.023	0	0	0.02	0	0	0	0.20

8.5 Comments

In this chapter, we have presented the following facts:

- To attain the convergence of suitability and optimality in portfolio selection an approach based on AHP and FMOP has been discussed.
- AHP technique has been used to calculate the local weights (performance scores) of each asset with respect to the four key asset allocation criteria. These weights have been used as objective function coefficients to formulate the multiobjective programming model.
- To solve the multiobjective programming model, an interactive fuzzy programming approach has been used.
- Using the computational results, it has been shown that the fuzzy interactive approach is very promising approach that can provide the preferred compromise solution which is indeed a compromise feasible solution that meets the investor preferences.

Chapter 9
Ethicality Considerations in Multi-criteria Fuzzy Portfolio Optimization

Abstract. Of late, the investors have shown great interest in socially responsible investment, also called ethical investment. Ideally, the investors may have a portfolio that is based not only on financial considerations but also incorporates a set of ethical values. The ethical investment movement that began from the USA in 1960s has gained tremendous momentum the world over recently. The growing instances of corporate scams and scandals have made it incumbent upon the investors to consider the quality of governance of corporations and ethicality of their conduct. Indeed, there has been a spate of reforms relating to corporate laws and capital markets all over the world. Also, the investors are becoming conscious of the desirability of ethical evaluation of the assets. The growing influence of institutional investors has reinforced this consciousness. The focus of this chapter is to present a comprehensive three-stage multiple criteria decision making framework for portfolio selection based upon financial and ethical criteria simultaneously. Fuzzy analytical hierarchy process (Fuzzy-AHP) technique is used to obtain the ethical performance score of each asset based upon investor preferences. A fuzzy multiple criteria decision making (Fuzzy-MCDM) method is used to obtain the financial quality score of each asset based upon investor-ratings on the financial criteria. Two hybrid portfolio optimization models are presented to obtain well diversified financially and ethically viable portfolios.

9.1 Ethical Evaluation of Assets

The framework for incorporation of ethicality in portfolio selection involves ethical screening of the assets and computation of their ethical scores.

9.1.1 Ethical Screening of Assets

One of the most important strategies applied by socially responsible investors is ethical screening of assets. According to such a strategy, the assets are

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screened on the basis of social, ethical and environmental considerations. Generally, two types of screening are used, namely, negative screening and positive screening [71]. Negative screening is the oldest and most basic socially responsible investment (SRI) filter. If a company is involved in businesses that are significantly detrimental to the given ethical issue(s); for example, the company is a tobacco manufacturer or dealer; then, the respective asset is excluded from further consideration. On the other hand, the positive screening involves the examination of whether a company is pursuing policies significantly in favor of the given ethical issue(s); for example, policies aimed at more inclusive workplaces; the respective asset is thus included for consideration in portfolio selection [5]. The following criteria are usually considered by investors all over for their presence or absence in ethical evaluation of assets:

Negative Screening Issues: firearms; weapons and military contracting; tobacco; gambling; human rights violation; child labour; oppressive regimes; pornography; alcohol; furs; excessive environmental impact and natural resources consumption; products dangerous to health/environment.

Positive Screening Issues: products beneficial for the environment and quality of life; product safety; environmental policies; management systems; employees policies, measures to avoid human rights violations; corporate governance.

The screening on the basis of aforesaid criteria leads to a set of ethical assets from which a desired portfolio may be constructed. Note that there could be assets that are neither significantly detrimental nor contributing toward an ethical issue. Such assets should be included for consideration in portfolio selection because they might contribute toward the financial performance of the portfolio. Their presence would also enhance the scope for portfolio diversification and the financial-ethical trade-off.

9.1.2 Ethical Performance Scores

In ethical evaluation of the financial assets, the rejection or selection of assets on the basis of negative and positive screening, respectively, is not enough. It is critical to score the assets with a view to discriminating among them on the basis of their comparative performance on the ethical criteria. Such an approach also permits the inclusion of the assets that are neither significantly negative nor positive. Further, in view of the growing interest in socially responsible investment, the investment advisors and experts are also in need of a systematic and comparative measurement of assets on ethical grounds rather than just relying on negative and positive screens. Toward this purpose, we define a measure called ethical performance (EP) score which can be used as an input along with the financial performance (FP) score in the portfolio selection models. The EP scores allow us to profile investor preferences for ethical considerations in respect of the assets to be included in the portfolio. Calculation of EP scores on the basis of perceptual data necessitates the recourse to a technique that permits systematic assessment of ethico-moral preferences of the investors. We use Fuzzy-AHP, a technique to aid decision making that involves a multi-level hierarchical structure for systematic decomposition of complex situations.

• **The hierarchical basis of ethical evaluation of the assets**

Determination of criteria/subcriteria for ethical evaluation of the assets is crucial for obtaining EP scores of the assets. There are several research & consulting organizations that provide regular information on these criteria. Note that, here, the criteria and subcriteria for ethical evaluation of the assets have been taken from the research presented in Gupta et al. [45]. The overall EP goal is decomposed into three main criteria of corporate social performance, namely, environmental sustainability (ES), corporate social responsibility (CSR) and corporate governance & business ethics (CGBE). Each of these criteria is further decomposed into three subcriteria apiece. The decomposition of each criterion into subcriteria is on the basis of those factors which are of prime concern that underline a company's performance on the said criterion. The reason of decomposing criteria into subcriteria lies in the fact that the criteria categories are too broad to be used directly for evaluating assets; hence, subcriteria within each criterion are considered. The resultant hierarchy is shown in Figure 9.1. Level 1 represents the goal, i.e., EP score; level 2 represents the three main criteria: ES, CSR and CGBE. At level 3, these criteria are decomposed into various subcriteria, i.e., ES is decomposed into emissions & waste disposal (EWD), resource conservation (RC), and recycling (RE); CSR is decomposed into product safety (PS), occupational safety (OS) and non-discrimination (ND); CGBE is decomposed into corruption (CR), disclosure (DI) and code of ethics (CE); and finally, the bottom level of the hierarchy, i.e., level 4 represents the alternatives (assets).

It is worthy to point out that the variables considered here represent just the illustrative elements of the ethical screen used for investment decision making rather than an exhaustive list of the subjective factors weighing in investor's mind at the time of decision making. It is also pertinent to mention that the variables affecting investment decision in this regard are not only multiple but are also diverse and dynamic in nature. That is, different investors would weigh these factors differently and also with time the dominant factors affecting ethical investing may undergo a change. We briefly discuss the variables considered here for modeling investment decisions involving ethical considerations.

• **Environmental Sustainability (ES):** It has become a buzz world for macroeconomic as well as microeconomic or firm-level decision making. The companies on their own and under pressure from the growing concern among investors and consumers alike are increasingly becoming conscious of their

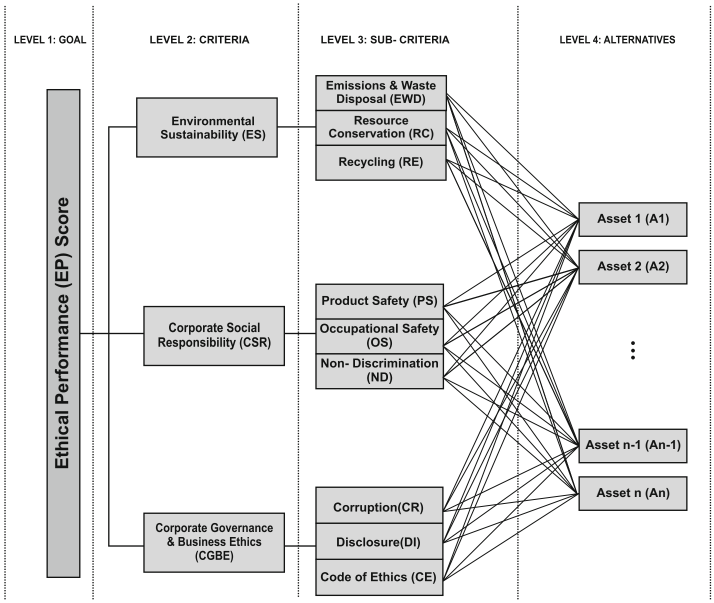

Fig. 9.1 Structural hierarchy for EP score

ecological footprint. Globally, investor surveys show that investing populace weighs corporate performance on this count while arriving at investment decision making. We have referred to three variables of prime concern that underline a company's performance on this count.

Emissions and Waste Disposal (EWD): Effluent treatment and disposal is obviously an important element of corporate performance on ecological front. Investors clearly weigh favorably those industries and those firms that demonstrate care and concern for EWD through their conduct.

Resource Conservation (RC): It relates to investor's concern regarding how positive are the companies on energy usage, water usage and general conservation of natural resources throughout their project and product life cycles.

Recycling (RE): The three R's (Reduce, Reuse and Recycle) represent the emerging ethos of sustainability and the extent to which a company is able to recycle inorganic inputs throughout its value chain and is thus an important contributory variable in its environmental performance.

• **Corporate Social Responsibility (CSR):** Corporate thinking on its responsibilities has over the years got fairly broad based from myopic focus on shareholders towards stakeholders in corporate performance. While the nature and relative importance of stakeholder may vary with the nature of industry and the bargaining power of the stakeholder, corporate performance or lack of it on product safety, occupational safety and non-discrimination is believed to contain a lot of price-sensitive information and has a bearing on investment decision making. We have referred to the following three variables that underline a company's performance on this count.

Product Safety (PS): Safety of the product or service for human/animal consumption is an important aspect of EP score. The boycott of companies selling tobacco, liquor, toys containing toxic colors/materials, etc. highlights the sensitivity of these issues.

Occupational Safety (OS): Certain industries such as mining are inherently risky for the employees; in other industries prolonged exposure to heat, dust, cold, chemicals, etc. expose the employees to several health and safety related hazards. Thus, how safe is the workplace becomes an important consideration in ethical investing.

Non-discrimination (ND): If fire, electrocution and exposure to heat, dust, cold, chemicals, etc. pose a physical hazard, discrimination on the grounds of sex, religion, caste, country or ethnic origin imply a social hazard. Socially inclusive workplaces find greater acceptance among the employees and the investors alike. Some companies are able to access capital markets much easily on account of their reputation for affirmative action on issues such as child labor, racial discrimination, etc.

- **Corporate Governance & Business Ethics (CGBE):** The frequency with which established corporates have tumbled and instances of corporate scams, scandals and misconduct have surfaced has brought the issue of corporate governance and business ethics on the top priority among the investors. The term corporate governance implies the mechanism for ensuring that the managements of the companies pursue the interests of the common shareholders rather than their own or that of the promoters alone. The three variables that we have considered here within the ambit of CGBE are explained as under.

Corruption (CR): If a company is known or perceived to be engaged in acts of corrupting public officials or the employees of the competitors, buyers, vendors, etc. or defrauding investors, consumers, etc. then investors would be wary of such a company as an investment avenue.

Disclosure (DI): Investment decision making is fraught not as much with the risk due to errors of judgment as with information asymmetry. How clearly and timely do the corporate reports communicate the realistic picture about the financial health and performance of the companies has a critical bearing on investment decisions.

Code of Ethics (CE): Code of ethics is a company's statement of what is perceived as morally right or wrong. It may pertain to such issues as bribery, prohibition of the company officers from trading on the stock market on the basis of information which only they come to know due to the position they occupy, sexual harassment at work place, etc. Investors' trust in the companies as regards the practice of ethics is an important determinant of investment choice in such companies.

• Computational procedure for obtaining relative preferences

The conventional AHP cannot fully capture subjective assessment of the various alternatives available to a decision maker. Fuzzy pair-wise comparisons are more rational to represent uncertain judgments than the crisp ones. By incorporating fuzzy set theory with AHP, Fuzzy-AHP use fuzzy numbers to evaluate pair-wise comparisons. The main steps of the Fuzzy-AHP procedure are described as under.

Step 1 : Construction of Hierarchy
A typical decision problem consists of : a number of alternatives, $m_i\,(i = 1, 2, \ldots, n)$, a set of evaluation criteria, $c_j\,(j = 1, 2, \ldots, m)$, a linguistic judgment r_{ij} representing the relative importance of each pair-wise comparison, and a weighting vector $w = (w_1, w_2, \ldots, w_n)$. For a decision problem, we first determine all important criteria, subcriteria and their relationships in the form of a hierarchy. The hierarchy is structured from the top (the overall goal of the problem) through the intermediate levels (criteria and subcriteria on which subsequent levels depend) to the bottom level (the list of alternatives).

Step 2 : Evaluation of Fuzzy Pair-Wise Comparisons
Once the hierarchy is established, pair-wise comparison evaluation takes place. All the criteria at the same level of the hierarchy are compared to each criterion at the preceding (upper) level using the linguistic terms [26] and the corresponding fuzzy numbers shown in Table 9.1.

Table 9.1 Linguistic scale and its fuzzy representation

Fuzzy number	Linguistic scales	Triangular Membership function
$\tilde{1}$	Equally important	(1,1,3)
$\tilde{3}$	Weakly important	(1,3,5)
$\tilde{5}$	Essentially important	(3,5,7)
$\tilde{7}$	Very strong important	(5,7,9)
$\tilde{9}$	Absolutely important	(7,9,9)

Fuzzy comparison matrix $\tilde{R}$ that represents fuzzy relative importance of each pair-wise comparison is obtained as

$$\tilde{R} = \begin{bmatrix} \tilde{1} & \tilde{r}_{12} & \ldots & \tilde{r}_{1m} \\ \tilde{r}_{21} & \tilde{1} & \ldots & \tilde{r}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{r}_{m1} & \tilde{r}_{m2} & \ldots & \tilde{1} \end{bmatrix} = \begin{bmatrix} \tilde{1} & \tilde{r}_{12} & \ldots & \tilde{r}_{1m} \\ \tilde{r}_{12}^{-1} & \tilde{1} & \ldots & \tilde{r}_{2m} \\ \ldots & \ldots & \ldots & \ldots \\ \tilde{r}_{1m}^{-1} & \tilde{r}_{2m}^{-1} & \ldots & \tilde{1} \end{bmatrix}$$

where

$$\tilde{r}_{ij} = \begin{cases} \tilde{1}, \tilde{3}, \tilde{5}, \tilde{7}, \tilde{9}, \text{ if the } i\text{-th criterion is relatively important to the } j\text{-th criterion}, \\ \tilde{1}, \text{ if } i = j, \\ \tilde{1}^{-1}, \tilde{3}^{-1}, \tilde{5}^{-1}, \tilde{7}^{-1}, \tilde{9}^{-1}, \text{ if the } i\text{-th criterion is relatively less important to the} \\ \qquad j\text{-th criterion}, \end{cases}$$

Step 3 : Calculating Fuzzy Weights and Checking Consistency
The fuzzy local weights can be calculated using Buckley's model [13] as follows:

$$\tilde{r}_i = [\tilde{r}_{i1} \otimes \tilde{r}_{i2} \ldots \otimes \tilde{r}_{im}]^{1/m}, \; i = 1, 2, \ldots, m,$$
$$\widetilde{w}_i = \frac{\tilde{r}_i}{\tilde{r}_1 \oplus \ldots \oplus \tilde{r}_m},$$

where $\tilde{r}_{ij}$ is the fuzzy comparison value of the i-th criterion to the j-th criterion, $\tilde{r}_i$ is the geometric mean of fuzzy comparison value of the i-th criterion to each other criterion, and $\tilde{w}_i$ is the fuzzy weight of the i-th criterion.

In order to control the result of the method, the consistency ratio (CR) needs to be calculated. The deviations from consistency are expressed by the following equation:

$$\text{Consistency index (CI)} = \frac{\lambda_{max} - m}{m - 1},$$

where m is the order of the paired comparison matrix. Since λ_{max} is a triangular fuzzy number, it is defuzzified into a crisp number to compute the CI. The central value of λ_{max} can be used for the purpose because of the symmetry of the triangular fuzzy number, the central value corresponds to the centroid of the triangular area.

The consistency ratio (CR) is calculated as

$$CR = CI/RI,$$

where RI is a known random consistency index that has been obtained from a large number of simulation runs and varies according to the order of matrix. If CI is sufficiently small then pair-wise comparisons are probably consistent enough to give useful estimates of the weights. The acceptable CR value for a matrix at each level is less than or equal to 0.1, i.e., if $CI/RI \leq 0.10$ then the degree of consistency is satisfactory; however, if $CI/RI > 0.10$ then serious inconsistencies may exist and hence Fuzzy-AHP may not yield meaningful results. The evaluation process should, therefore, be reviewed and improved. The eigenvectors are used to calculate the global weights if there is an acceptable degree of consistency of the selection criteria.

Step 4 : Hierarchical Layer Sequencing
The final fuzzy weight value $\widetilde{U}_i = (l_i, m_i, u_i)$ of the i-th alternative is calculated by hierarchical layer sequencing as

$$\widetilde{U}_i = \sum_{j=1}^{m} \tilde{w}_j \tilde{e}_{ij} ,$$

where $\tilde{e}_{ij}$ is the fuzzy weight value of the j-th criterion to the i-th alternative obtained using pair-wise comparisons.

Step 5 : Ranking Alternatives
Since $\widetilde{U}_i$ are fuzzy numbers, a defuzzification method is used for obtaining a crisp value of the fuzzy number in order to choose the optimum alternative. There are number of procedures to perform the ranking of fuzzy numbers [12]. Among them, we use the representative method in which the following relation is employed.

$$R(\widetilde{U}_i) = \frac{l_i + 2m_i + u_i}{4},$$

where $R(\widetilde{U}_i)$ represents the representative ordinal of a triangular fuzzy number.

9.2 Financial Evaluation of Assets

The financial quality of the assets is measured in terms of their potential short and long term returns, liquidity and risk related characteristics. An estimation of these characteristic by extrapolation of historical data is fraught with the possibility of measurement and judgmental errors as asset performance is contingent upon a host of environmental and market related factors. Moreover, the investors are more comfortable in articulating their preferences only linguistically such as high return, low risk. The vagueness in such expressions necessitate recourse to fuzzy methodology for determining the financial quality of the assets under their contemplation. Thus, we employ a Fuzzy-MCDM method for determining the overall financial quality score of each asset with respect to the financial criteria. In the following discussion, we present details of the fuzzy-MCDM method developed by Lee [79]. We first present some basic definitions and concepts as introduced in Bector and Chandra [6]; Zimmermann [130].

Definition 9.1 (Fuzzy number). Fuzzy set $\tilde{A}$ in $X \subset R$, the set of real numbers, is a set of ordered pairs $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where x is the generic element of X and $\mu_{\tilde{A}}(x)$ is the membership function or grade of membership, or degree of compatibility or degree of truth of $x \in X$ which maps $x \in X$ on the interval $[0, 1]$.

Definition 9.2 (α-level cut). Let $\tilde{A}$ be a fuzzy set in X. The crisp set A_α of elements that belong to the fuzzy set $\tilde{A}$ at least to the degree $\alpha \in [0, 1]$ is called the α-level cut (α-level set) of fuzzy set $\tilde{A}$ and is given by $A_\alpha = \{x \in X | \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 9.3 (Lower and upper α-level cuts). If $\tilde{A}$ be a fuzzy number, the α-level sets A_α can be written as $A_\alpha = [A^L_\alpha, A^R_\alpha]$. A^L_α and A^U_α are called lower and upper α-level cuts and are defined as $A^L_\alpha = \inf_{\mu_{\tilde{A}}(x)\geq\alpha}(x)$ and $A^U_\alpha = \sup_{\mu_{\tilde{A}}(x)\geq\alpha}(x)$, respectively. Here, inf and sup are used to find the minimum and maximum elements of the α-level cuts, respectively.

Definition 9.4 (Extended fuzzy preference relation). For two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the extended fuzzy preference relation $F(\tilde{A},\tilde{B})$ is defined by the membership function

$$\mu_F(\tilde{A},\tilde{B}) = \int_0^1 ((\tilde{A}-\tilde{B})^L_\alpha + (\tilde{A}-\tilde{B})^U_\alpha) d\alpha \tag{9.1}$$

Remark 9.1. *If $\tilde{A} = (l_1, m_1, n_1)$ and $\tilde{B} = (l_2, m_2, n_2)$ are two triangular fuzzy numbers then*

$$\mu_F(\tilde{A},\tilde{B}) = (l_1 + 2m_1 + n_1 - l_2 - 2m_2 - n_2)/2.$$

It may be noted that $\mu_F(\tilde{A},\tilde{B}) \geq 0$ if and only if $(l_1+2m_1+n_1-l_2-2m_2-n_2) \geq 0$.

The following proposition provides some important properties of the extended fuzzy preference relation.

Proposition 9.1. *For any three fuzzy numbers $\tilde{A}, \tilde{B}$ and $\tilde{C}$, the following statements hold true for the extended fuzzy preference relation F:*

(i) F is reciprocal, i.e., $\mu_F(\tilde{B},\tilde{A}) = -\mu_F(\tilde{A},\tilde{B})$.
(ii) F is additive, i.e., $\mu_F(\tilde{A},\tilde{B}) + \mu_F(\tilde{B},\tilde{C}) = \mu_F(\tilde{A},\tilde{C})$.
(iii) F is transitive, i.e., $\mu_F(\tilde{A},\tilde{B}) \geq 0$ and $\mu_F(\tilde{B},\tilde{C}) \geq 0 \Rightarrow \mu_F(\tilde{A},\tilde{C}) \geq 0$.

Definition 9.5 (Preference intensity function). The preference intensity function of one fuzzy number $\tilde{A}$ over another fuzzy number $\tilde{B}$ is defined as

$$Q(\tilde{A},\tilde{B}) = \begin{cases} \mu_F(\tilde{A},\tilde{B}), & \text{if } \mu_F(\tilde{A},\tilde{B}) \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{9.2}$$

Further, the following operational laws of two triangular fuzzy numbers $\tilde{A}_1 = (l_1, m_1, u_1)$ and $\tilde{A}_2 = (l_2, m_2, u_2)$ hold true :

- Fuzzy number addition

$$\tilde{A}_1 \oplus \tilde{A}_2 = (l_1, m_1, u_1) \oplus (l_2, m_2, u_2) = (l_1 + l_2, m_1 + m_2, u_1 + u_2).$$

- Fuzzy number multiplication

$$\tilde{A}_1 \otimes \tilde{A}_2 = (l_1, m_1, u_1) \otimes (l_2, m_2, u_2) = (l_1 \times l_2, m_1 \times m_2, u_1 \times u_2) \quad \text{for } l_1, l_2, m_1, m_2, u_1, u_2 > 0.$$

- Fuzzy number division

$$\tilde{A}_1 \oslash \tilde{A}_2 = (l_1, m_1, u_1) \oslash (l_2, m_2, u_2) = (l_1/u_2, m_1/m_2, u_1/l_2) \quad \text{for } l_1, l_2, m_1, m_2, u_1, u_2 > 0.$$

- Fuzzy number reciprocal

$$\tilde{A}^{-1} = (l, m, u)^{-1} = (1/u, 1/m, 1/l) \text{ for } l, m, u > 0.$$

Since 'quality' is a subjective phenomenon, therefore, any discussion of asset quality must factor in investor preferences. For example, investors differ in terms of their relative preferences for short term versus long term returns, liquidity over risk or risk over return. Accordingly, each investor would typically (a) show distinct preference for the asset allocation criteria and (b) indicate the perceived quality of the various assets on each criterion. From this information, it becomes possible to measure the advantage and disadvantage of each asset over the rest on each of the financial criteria used. The advantage score of the asset when multiplied by the weight of each criterion leads to a measure of the asset's fuzzy strength whereas the disadvantage score of the asset when multiplied by the weight of each criterion leads to a measure of its fuzzy weakness. The FP score of the asset is an aggregate of the asset's relative performance derived from its fuzzy strength as well as its fuzzy weakness. Note that by the strength of the asset based on fuzzy weakness, we mean how much an asset is preferred over the other assets based on the fuzzy weakness scores and thus, this preference could also be considered as its strength. It may be possible that an asset does not perform well in comparison with some other asset(s) on the basis of its fuzzy strength score; but, after combining its fuzzy weakness score as well, the performance of the asset may become more acceptable.

We assume that there are n assets under evaluation against m criteria. Let the indices i and k denotes the assets under consideration and the index j denotes the evaluation criteria. Let fuzzy number $\tilde{A}_{ij}$ be rating of the i-th asset on the j-th criterion and fuzzy number $\tilde{w}_j$ be the weight of the j-th criterion. Let J be the set of benefit criteria (i.e., larger the value is, the better the asset is) and J' be the set of negative criteria (i.e., smaller the value is, the better the asset is) with $J \cup J' = \{1, \ldots, m\}$ and $J \cap J' = \emptyset$.

The crisp advantage of the i-th asset relative to all other assets $k \neq i$ on the j-th criterion is given as

$$a_{ij} = \begin{cases} \sum\limits_{k \neq i} Q(\tilde{A}_{ij}, \tilde{A}_{kj}), & \text{if } j \in J, \\ \sum\limits_{k \neq i} Q(\tilde{A}_{kj}, \tilde{A}_{ij}), & \text{if } j \in J'. \end{cases} \tag{9.3}$$

Similarly, the crisp disadvantage of the i-th asset relative to all other assets $k \neq i$ on the j-th criterion is given as

$$d_{ij} = \begin{cases} \sum_{k \neq i} Q(\tilde{A}_{kj}, \tilde{A}_{ij}), & \text{if} \quad j \in J, \\ \sum_{k \neq i} Q(\tilde{A}_{ij}, \tilde{A}_{kj}), & \text{if} \quad j \in J'. \end{cases} \tag{9.4}$$

The fuzzy strength of the i-th asset using its crisp advantage on all the m evaluation criteria is now obtained as

$$FS_i = \sum_{j=1}^{m} a_{ij} \tilde{w}_j, \tag{9.5}$$

and the fuzzy weakness of the i-th asset using its crisp disadvantage on all the m evaluation criteria is now obtained as

$$FW_i = \sum_{j=1}^{m} d_{ij} \tilde{w}_j. \tag{9.6}$$

The crisp strength of the i-th asset using its fuzzy strength and fuzzy weakness relative to all other assets $k \neq i$ is obtained as

$$S_i = \sum_{k \neq i} Q(FS_i, FS_k) + \sum_{k \neq i} Q(FW_k, FW_i) \tag{9.7}$$

and the crisp weakness of the i-th asset using its fuzzy strength and fuzzy weakness relative to all other assets $k \neq i$ is obtained as

$$I_i = \sum_{k \neq i} Q(FS_k, FS_i) + \sum_{k \neq i} Q(FW_i, FW_k). \tag{9.8}$$

The FP score of the i-th asset is now obtained as

$$T_i = \frac{S_i}{S_i + I_i}, \tag{9.9}$$

and its normalized value is obtained as

$$T_i' = \frac{T_i}{\sum_{i=1}^{n} T_i}. \tag{9.10}$$

9.3 Hybrid Portfolio Selection Models

We assume that the investor allocate his/her wealth among n assets. The following notation are used in the formulation of the portfolio selection model.

9.3.1 Notation

f_i: the normalized FP score of the i-th asset calculated using Fuzzy-MCDM method ,

e_i: the normalized EP score of the i-th asset calculated using Fuzzy-AHP ,

x_i: the proportion of the total funds invested in the i-th asset ,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

u_i: the maximal fraction of the capital allocated to the i-th asset ,

l_i: the minimal fraction of the capital allocated to the i-th asset .

We consider the following objective function and constraints in the portfolio selection problem.

9.3.2 Objective Function

Financial Criteria
The objective function using FP scores based on the four key financial criteria is expressed as

$$Z(x) = \sum_{i=1}^{n} f_i x_i.$$

9.3.3 Constraints

Ethical Investing Constraint
When the investor chooses the desired ethical level of the portfolio *a priori*, an ethical investing constraint may be imposed on the portfolio selection. The ethical investing constraint using the EP scores is expressed as

$$\sum_{i=1}^{n} e_i x_i \geq \beta ,$$

where $0 \leq \beta \leq \max_{1\leq i\leq n} e_i$ is regarded as investor's choice for a minimum desired ethical level of the portfolio. The following cases may arise:

(i) if $\beta > \max_{1\leq i\leq n} e_i$, then, no feasible solution can be found and hence no portfolio is generated,
(ii) if $\beta = \max_{1\leq i\leq n} e_i$, then, there is only one portfolio corresponding to $x_p = 1$ where $e_p = \max_{1\leq i\leq n} e_i$, i.e., the p-th asset has the maximum EP score,
(iii) if $0 \leq \beta \leq \max_{1\leq i\leq n} e_i$, then, the higher the β-value is, i.e., closer to $\max_{1\leq i\leq n} e_i$, the higher the impact of desired ethical level in the portfolio construction. The lower the β-value is, i.e., closer to 0, the lower the impact of desired ethical level in the portfolio construction.

Entropy Constraint
In portfolio selection, the more uniformly the capital is allocated to all the assets, the more diverse the investment is. A portfolio allocation among n assets, with properties $x_i \geq 0\,, i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} x_i = 1$, has the structure of a proper probability distribution. To handle the issue of diversification, we use the following concave entropy function proposed by Shannon [109] as a measure of portfolio diversification which is defined as

$$E(x) = -\sum_{i=1}^{n} x_i \ln x_i \,.$$

It may be noted that when $x_i = 1/n$, $i = 1, 2, \ldots, n$, $E(x)$ has its maximum value $\ln n$. The other extreme case occurs when $x_i = 1$ for one i, and $x_i = 0$ for the rest, then $E(x) = 0$. Therefore, entropy that provides a good measure of disorder in a system or expected information in a probability distribution, can be taken as a measure of portfolio diversification. In order to achieve desired level of portfolio diversification, we use the following constraint

$$-\sum_{i=1}^{n} x_i \ln x_i \geq \gamma\,,\ \ 0 < \gamma < \ln n\,,$$

where gamma (γ) is the preset entropy value given by the investor.

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1\,.$$

No short selling of assets is expressed as

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,.$$

9.3.4 The Decision Problem

The hybrid portfolio selection problem is now formulated as follows:

$$\text{P}(9.1) \quad \max Z(x) = \sum_{i=1}^{n} f_i x_i$$

subject to

$$\sum_{i=1}^{n} x_i = 1\,, \tag{9.11}$$

$$x_i \geq 0\,, \quad i = 1, 2, \ldots, n\,, \tag{9.12}$$

$$\sum_{i=1}^{n} e_i x_i \geq \beta\,, \tag{9.13}$$

$$-\sum_{i=1}^{n} x_i \ln x_i \geq \gamma\,. \tag{9.14}$$

Proposition 9.2. *The optimization problem P(9.1) is a convex optimization problem, i.e., it involves the maximization of a linear function subject to convex constraints.*

Proof. The objective function $\sum_{i=1}^{n} f_i x_i$ of the problem P(9.1) is a linear function which is both convex and concave. The constraint (9.11) is a linear function and hence it is convex. The constraint (9.12) is also a linear function and it can be rewritten as $-x_i \leq 0, \quad i = 1, 2, \ldots, n$, hence, it can be taken as convex. The constraint (9.13) is a linear function and can be rewritten as $-\sum_{i=1}^{n} e_i x_i + \beta \leq 0$, hence, it can also be taken as convex function. It may be noted that the entropy function $E(x) = -\sum_{i=1}^{n} x_i \ln x_i$ is a concave function as mentioned above. Therefore, $-E(x) = \sum_{i=1}^{n} x_i \ln x_i$ is a convex function and thus the constraint (9.14) which can be rewritten as $\sum_{i=1}^{n} x_i \ln x_i + \gamma \leq 0,\ 0 < \gamma < \ln n$ is also convex. Hence, the optimization problem P(9.1) which involves the maximization of a linear function subject to convex constraints is a convex optimization problem. □

Note that since the problem P(9.1) is a convex optimization problem, therefore, a local max point of P(9.1) is also its global max point.

It is worthy to point out that unlike the model P(9.1), we can also handle the issue of portfolio diversification by using the following constraints:

- *Maximal fraction of the capital that can be invested in a single asset is expressed as*

$$x_i \leq u_i y_i\,, \quad i = 1, 2, \ldots, n\,. \tag{9.15}$$

- *Minimal fraction of the capital that can be invested in a single asset is expressed as*

$$x_i \geq l_i y_i\,, \quad i = 1, 2, \ldots, n\,. \tag{9.16}$$

- *Number of assets held in the portfolio is expressed as*

$$\sum_{i=1}^{n} y_i = h\,. \tag{9.17}$$

- *Selection or rejection of assets is expressed as*

$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,. \tag{9.18}$$

Thus, the hybrid portfolio selection problem may be formulated as the following mixed integer linear programming problem:

P(9.2) $\max Z(x) = \sum_{i=1}^{n} f_i x_i$

subject to

Constraints (9.11-9.13) and (9.15-9.18).

9.4 Numerical Illustration

In this section, we present the results of an empirical study done for an imaginary socially responsible investor. We have randomly selected 15 assets listed on NSE, Mumbai, India, for numerical illustrations.

9.4.1 Ethical Screening and Ethical Performance Scores

By applying negative screens, discussed in Section 9.1, we exclude 5 assets from the population. For the remaining 10 assets, we calculate the EP scores using Fuzzy-AHP. The procedure followed is a pair-wise comparison of the criteria, subcriteria and the assets. For the data in respect of pair-wise comparison matrices, we have relied on inputs from the imaginary investor that are based on the linguistic scale provided in Table 9.1. The computations of the Fuzzy-AHP procedure are described as follows: At level 2, we determine the local weights (see Table 9.2) of the three main criteria with respect to the overall goal of EP score. At level 3, we determine local weights (see Table 9.3) of the various subcriteria with respect to their parent criterion in the level 2. For example, the subcriteria, RC, RE and EWD are pair-wise compared with respect to the parent criterion ES. At level 4, we determine the local weights (see Tables 9.4-9.6) of all the 10 assets with respect to each of the nine subcriteria of ethical evaluation in the level 3.

Table 9.2 Pair-wise comparisons of the main criteria in relation to the overall goal

Criteria	ES	CSR	CGBE	Local weight
ES	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	(0.0374, 0.1047, 0.4287)
CSR	$\tilde{5}$	$\tilde{1}$	$\tilde{3}$	(0.1762, 0.6370, 2.0224)
CGBE	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	(0.0715, 0.2583, 1.0572

Table 9.3 Pair-wise comparisons of the subcriteria in relation to the main criteria

ES	RC	RE	EWD	Local weight
RC	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	(0.0333, 0.0879, 0.3274)
RE	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.0693, 0.2426, 0.9574)
EWD	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	(0.2026, 0.6694, 1.9915)
CSR	ND	OS	PS	Local weight
ND	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	(0.0374, 0.1047, 0.4287)
OS	$\tilde{5}$	$\tilde{1}$	$\tilde{3}$	(0.1762, 0.6370, 2.0224)
PS	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	(0.0715, 0.2583, 1.0572)
CGBE	CR	CE	DI	Local weight
CR	$\tilde{1}$	$\tilde{7}^{-1}$	$\tilde{9}^{-1}$	(0.0270, 0.0549, 0.1402)
CE	$\tilde{7}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.1167, 0.2897, 0.9542)
DI	$\tilde{9}$	$\tilde{3}$	$\tilde{1}$	(0.2232, 0.6554, 1.6316)

These local weights are aggregated in respect of each asset by following what in terms of the AHP hierarchy may be regarded as a bottom-up process of successive multiplication. The local weight of an asset in relation to a subcriterion is multiplied with the local weight of the subcriterion in relation to its parent criterion, which in turn is multiplied with the local weight of the parent criterion in relation to the overall goal of EP score. Thus, we obtain 9 aggregated local weights for each asset. The global weight of an asset in relation to each main criterion involving all its subcriteria is obtained by adding the aggregated local weights of the asset in relation to the said criterion through its subcriteria (columns 2, 3 and 4 of the Table 9.7 presents the global weights of the assets in respect of the three main criteria). To calculate the EP score, the global weights of each asset are summed. The fuzzy EP scores of the 10 assets are listed in column 5 of the Table 9.7. Finally, using defuzzification process, the crisp EP scores and the normalized EP scores of the 10 assets are obtained, see columns 6 and 7 of the Table 9.7.

Table 9.4 Pair-wise comparisons of the alternatives in relation to the subcriteria RC, RE and EWD

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Local weight
RC											
A1	$\tilde{1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{1}$	(0.06080, 0.15618, 0.48673)
A2	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{7}^{-1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{3}$	$\tilde{3}^{-1}$	(0.02337, 0.07670, 0.26272)
A3	$\tilde{3}$	$\tilde{7}$	$\tilde{1}$	$\tilde{9}$	$\tilde{5}$	$\tilde{5}$	$\tilde{7}$	$\tilde{9}$	$\tilde{7}$	$\tilde{3}$	(0.11970, 0.32046, 0.75532)
A4	$\tilde{9}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	(0.00759, 0.01747, 0.05261)
A5	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.01737, 0.04580, 0.19042)
A6	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{1}$	(0.04563, 0.12966, 0.33263)
A7	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{7}^{-1}$	(0.01338, 0.02949, 0.10003)
A8	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{7}^{-1}$	(0.00971, 0.02689, 0.07377)
A9	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01372, 0.03582, 0.10266)
A10	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{7}$	$\tilde{7}$	$\tilde{5}$	$\tilde{1}$	(0.05136, 0.16152, 0.40066)
RE											
A1	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{3}$	$\tilde{5}$	$\tilde{3}^{-1}$	(0.02258, 0.06542, 0.19909)
A2	$\tilde{5}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{9}$	$\tilde{7}$	$\tilde{7}$	$\tilde{9}$	$\tilde{3}$	(0.08894, 0.22671, 0.54875)
A3	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{7}$	$\tilde{7}$	$\tilde{9}$	$\tilde{9}$	$\tilde{3}$	(0.10995, 0.29209, 0.64458)
A4	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	(0.00614, 0.01185, 0.03229)
A5	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{1}$	$\tilde{7}$	$\tilde{5}$	$\tilde{7}$	$\tilde{7}$	$\tilde{3}^{-1}$	(0.05131, 0.12765, 0.37502)
A6	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	(0.01168, 0.02856, 0.07910)
A7	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	(0.01303, 0.03029, 0.09609)
A8	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{9}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{7}^{-1}$	(0.01493, 0.03971, 0.10913)
A9	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.00818, 0.01871, 0.05936)
A10	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{7}$	$\tilde{7}$	$\tilde{7}$	$\tilde{5}$	$\tilde{1}$	(0.06027, 0.15901, 0.44051)
EWD											
A1	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{5}$	$\tilde{5}$	$\tilde{5}^{-1}$	(0.02369, 0.06173, 0.19973)
A2	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{7}$	$\tilde{7}$	$\tilde{9}$	$\tilde{3}^{-1}$	(0.04865, 0.12667, 0.36377)
A3	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{9}$	$\tilde{9}$	$\tilde{9}$	$\tilde{9}$	$\tilde{1}$	(0.11435, 0.27100, 0.61443)
A4	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	(0.00726, 0.01582, 0.04392)
A5	$\tilde{3}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{1}$	$\tilde{7}$	$\tilde{7}$	$\tilde{7}$	$\tilde{7}$	$\tilde{3}^{-1}$	(0.06491, 0.16749, 0.45316)
A6	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{7}^{-1}$	(0.01369, 0.03534, 0.09012)
A7	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{9}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{9}^{-1}$	(0.01528, 0.03534, 0.10059)
A8	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	$\tilde{9}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	(0.00912, 0.01988, 0.06052)
A9	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.00984, 0.02373, 0.06995)
A10	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{7}$	$\tilde{9}$	$\tilde{7}$	$\tilde{5}$	$\tilde{1}$	(0.08801, 0.24300, 0.53684)

Table 9.5 Pair-wise comparisons of the alternatives in relation to the subcriteria ND, OS and PS

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Local weight
ND											
A1	$\tilde{1}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	(0.01288, 0.03266, 0.13233)
A2	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	$\tilde{7}^{-1}$	(0.00603, 0.01288, 0.03535)
A3	$\tilde{3}$	$\tilde{9}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	(0.02712, 0.06481, 0.27396)
A4	$\tilde{3}$	$\tilde{5}$	$\tilde{1}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	(0.02600, 0.06873, 0.26716)
A5	$\tilde{5}$	$\tilde{9}$	$\tilde{3}$	$\tilde{5}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{3}$	$\tilde{3}$	(0.05806, 0.18192, 0.57692)
A6	$\tilde{5}$	$\tilde{9}$	$\tilde{3}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{3}$	$\tilde{3}$	(0.04661, 0.17286, 0.49979)
A7	$\tilde{7}$	$\tilde{7}$	$\tilde{3}$	$\tilde{5}$	$\tilde{3}$	$\tilde{3}$	$\tilde{1}$	$\tilde{7}$	$\tilde{3}$	$\tilde{5}$	(0.08152, 0.27765, 0.77560)
A8	$\tilde{1}^{-1}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.01116, 0.03104, 0.10623)
A9	$\tilde{3}$	$\tilde{7}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	(0.02349, 0.07873, 0.27396)
A10	$\tilde{3}$	$\tilde{7}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{1}$	(0.02272, 0.07873, 0.22744)
OS											
A1	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.02156, 0.06138, 0.19525)
A2	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{9}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{5}$	$\tilde{5}$	$\tilde{3}^{-1}$	(0.04938, 0.13487, 0.40064)
A3	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	$\tilde{9}$	$\tilde{7}$	$\tilde{9}$	$\tilde{9}$	$\tilde{5}$	$\tilde{5}$	$\tilde{1}$	(0.12428, 0.26295, 0.64248)
A4	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	(0.00677, 0.01374, 0.03847)
A5	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{7}^{-1}$	$\tilde{9}$	$\tilde{1}$	$\tilde{3}$	$\tilde{7}$	$\tilde{7}$	$\tilde{5}$	$\tilde{3}^{-1}$	(0.04620, 0.13254, 0.32135)
A6	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{5}^{-1}$	(0.01731, 0.05095, 0.15541)
A7	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.00974, 0.02568, 0.07698)
A8	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	(0.01173, 0.02867, 0.08378)
A9	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	$\tilde{5}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01672, 0.04338, 0.13012)
A10	$\tilde{5}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{9}$	$\tilde{3}$	$\tilde{5}$	$\tilde{5}$	$\tilde{9}$	$\tilde{5}$	$\tilde{1}$	(0.08276, 0.24584, 0.54322)
PS											
A1	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.01413, 0.03371, 0.15673)
A2	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	(0.02951, 0.09299, 0.39115)
A3	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	$\tilde{7}$	$\tilde{7}$	$\tilde{9}$	$\tilde{3}$	$\tilde{3}$	$\tilde{7}$	$\tilde{3}$	(0.09417, 0.28158, 0.80195)
A4	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{7}^{-1}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01155, 0.02943, 0.10711)
A5	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{7}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	(0.02493, 0.08543, 0.29836)
A6	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{7}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	(0.00911, 0.02706, 0.09279)
A7	$\tilde{5}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{7}$	$\tilde{3}$	$\tilde{7}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{3}$	(0.05958, 0.21133, 0.64428)
A8	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	(0.02951, 0.10299, 0.33585)
A9	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.01135, 0.03762, 0.13241)
A10	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{1}$	(0.02369, 0.09786, 0.29095)

Table 9.6 Pair-wise comparisons of the alternatives with respect to the subcriteria CR, CE and DI

	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	Local weight
CR											
A1	$\tilde{1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.02381, 0.06338, 0.22220)
A2	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{7}^{-1}$	(0.00955, 0.02150, 0.07909)
A3	$\tilde{5}$	$\tilde{9}$	$\tilde{1}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	(0.10736, 0.22011, 0.60750)
A4	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	(0.01977, 0.06670, 0.19908)
A5	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	(0.00827, 0.02043, 0.06349)
A6	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	(0.02063, 0.06022, 0.17837)
A7	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	$\tilde{9}^{-1}$	(0.01122, 0.02989, 0.09777)
A8	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{1}^{-1}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	(0.01848, 0.06338, 0.14808)
A9	$\tilde{3}$	$\tilde{9}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{9}$	$\tilde{5}$	$\tilde{9}$	$\tilde{5}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.06681, 0.18723, 0.48353)
A10	$\tilde{5}$	$\tilde{7}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{9}$	$\tilde{5}$	$\tilde{9}$	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	(0.08913, 0.26717, 0.58740)
CE											
A1	$\tilde{1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{5}$	$\tilde{3}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.03024, 0.09834, 0.31234)
A2	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{9}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01498, 0.03278, 0.12834)
A3	$\tilde{5}$	$\tilde{9}$	$\tilde{1}$	$\tilde{7}$	$\tilde{7}$	$\tilde{9}$	$\tilde{3}$	$\tilde{9}$	$\tilde{5}$	$\tilde{3}$	(0.14840, 0.30837, 0.70838)
A4	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.01576, 0.03752, 0.13273)
A5	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{7}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{7}^{-1}$	(0.01806, 0.04519, 0.14814)
A6	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{9}^{-1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01659, 0.04558, 0.13506)
A7	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{7}^{-1}$	(0.02051, 0.07253, 0.17267)
A8	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{9}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{5}^{-1}$	(0.01366, 0.04331, 0.11410)
A9	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{5}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.01884, 0.07073, 0.14336)
A10	$\tilde{3}$	$\tilde{5}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{7}$	$\tilde{5}$	$\tilde{5}$	$\tilde{1}$	(0.08779, 0.24564, 0.60357)
DI											
A1	$\tilde{1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{1}$	$\tilde{3}^{-1}$	(0.03778, 0.09353, 0.38373)
A2	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{5}^{-1}$	(0.01841, 0.06121, 0.20713)
A3	$\tilde{3}$	$\tilde{5}$	$\tilde{1}$	$\tilde{7}$	$\tilde{3}$	$\tilde{3}$	$\tilde{7}$	$\tilde{7}$	$\tilde{3}$	$\tilde{1}$	(0.07569, 0.21050, 0.65396)
A4	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	(0.00831, 0.02275, 0.07436)
A5	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{5}^{-1}$	(0.02354, 0.08445, 0.23909)
A6	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}$	$\tilde{1}$	$\tilde{1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	(0.02072, 0.06779, 0.24518)
A7	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{3}$	$\tilde{1}^{-1}$	$\tilde{1}^{-1}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}^{-1}$	$\tilde{7}^{-1}$	(0.01291, 0.04365, 0.12781)
A8	$\tilde{5}^{-1}$	$\tilde{3}^{-1}$	$\tilde{7}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{3}^{-1}$	$\tilde{1}$	$\tilde{3}^{-1}$	$\tilde{9}^{-1}$	(0.00993, 0.02457, 0.09997)
A9	$\tilde{1}^{-1}$	$\tilde{3}$	$\tilde{3}^{-1}$	$\tilde{5}$	$\tilde{1}$	$\tilde{3}$	$\tilde{5}$	$\tilde{3}$	$\tilde{1}$	$\tilde{5}^{-1}$	(0.03273, 0.11071, 0.34117)
A10	$\tilde{3}$	$\tilde{5}$	$\tilde{1}^{-1}$	$\tilde{7}$	$\tilde{5}$	$\tilde{3}$	$\tilde{7}$	$\tilde{9}$	$\tilde{5}$	$\tilde{1}$	(0.08737, 0.28083, 0.68212)

Table 9.7 Global weights of the assets

	Global weight					
Assets	ES	CSR	CGBE	Fuzzy EP score	Defuzzified EP score	Normalized EP score
A1	(0.0003, 0.0074, 0.3205)	(0.0009, 0.0326, 1.2484)	(0.0009, 0.0241, 1.0100)	(0.0021 0.0641 2.5789)	0.6773	0.0792
A2	(0.0006, 0.0153, 0.5726)	(0.0019, 0.0709, 2.5055)	(0.0004, 0.0131, 0.4985)	(0.0030, 0.0993, 3.5767)	0.9446	0.1104
A3	(0.0013, 0.0294, 0.8951)	(0.0052, 0.1573, 4.5798)	(0.0027, 0.0618, 1.9327)	(0.0092, 0.2485, 7.4076)	1.9785	0.2313
A4	(0.0001, 0.0016, 0.0581)	(0.0005, 0.0150, 0.6180)	(0.0003, 0.0076, 0.2917)	(0.0009, 0.0242, 0.9678)	0.2543	0.0297
A5	(0.0006, 0.0154, 0.5675)	(0.0021, 0.0800, 2.4523)	(0.0005, 0.0180, 0.5713)	(0.0033, 0.1133, 3.5911)	0.9553	0.1117
A6	(0.0002, 0.0044, 0.1561)	(0.0010, 0.0367, 1.2673)	(0.0005, 0.0157, 0.5856)	(0.0017, 0.0568, 2.0090)	0.5311	0.0621
A7	(0.0002, 0.0035, 0.1393)	(0.0016, 0.0637, 2.3647)	(0.0004, 0.0132, 0.4091)	(0.0022, 0.0805, 2.9132)	0.7691	0.0899
A8	(0.0001, 0.0027, 0.1068)	(0.0008, 0.0306, 1.1528)	(0.0003, 0.0083, 0.3095)	(0.0012, 0.0416, 1.5691)	0.4134	0.0483
A9	(0.0001, 0.0025, 0.0985)	(0.0008, 0.0290, 1.0528)	(0.0008, 0.0267, 0.8048)	(0.0017, 0.0582, 1.9561)	0.5186	0.0606
A10	(0.0009, 0.0226, 0.6953)	(0.0030, 0.1211, 3.0410)	(0.0023, 0.0697, 1.8726)	(0.0062, 0.2134, 5.6089)	1.5105	0.1766

9.4.2 Financial Performance Scores

To obtain financial scores of the assets, we use the following four evaluation criteria:

Short term return (C_1); Long term return (C_2); Risk (C_3); Liquidity (C_4).

Here, C_1, C_2 and C_4 are benefit criteria, whereas C_3 is a negative criterion. The data to evaluate the financial performance of the assets in respect of the four criteria can be obtained from the inputs of the imaginary investor. The investor preferences were captured using the linguistic variables employed to represent relative importance and ratings provided in Tables 9.8-9.9, respectively. The data in respect of the weights of these criteria and rating of the assets are shown in Tables 9.10 and 9.11, respectively.

Table 9.8 Linguistic variables for the relative importance of the criteria

Linguistic variables	Fuzzy number
Very low (VL)	(0, 0, 0.1)
Low (L)	(0, 0.1, 0.3)
Medium low (ML)	(0.1, 0.3, 0.5)
Medium (M)	(0.3, 0.5, 0.7)
Medium high (MH)	(0.5, 0.7, 0.9)
High (H)	(0.7, 0.9, 1.0)
Very high (VH)	(0.9, 1.0, 1.0)

Table 9.9 Linguistic variables for the performance ratings

Linguistic variables	Fuzzy number
Very poor (VP)	(0, 0, 1)
Poor (P)	(0, 1, 3)
Medium poor (MP)	(1, 3, 5)
Fair (F)	(3, 5, 7)
Medium good (MG)	(5, 7, 9)
Good (G)	(7, 9, 10)
Very good (VG)	(9, 10, 10)

Table 9.10 The weights of the evaluation criteria

	C_1	C_2	C_3	C_4
Weight	(0.5, 0.7, 0.9)	(0.9, 1.0, 1.0)	(0.7, 0.9, 1.0)	(0.3, 0.5,0.7)

The above data is processed using the Fuzzy-MCDM method. The evaluation procedure followed to arrive at the FP score of each asset is as per the description given in Section 9.2. The corresponding computational results are listed in Tables 9.12-9.13. Table 9.14 presents the FP score and its normalized value for each asset.

Table 9.11 The ratings of the assets

	Fuzzy number				Normalized fuzzy number			
Assets	C_1	C_2	C_3	C_4	C_1	C_2	C_3	C_4
A1	(7, 9, 10)	(5, 7, 9)	(7, 9, 10)	(3, 5, 7)	(0.7, 0.9, 1.0)	(0.5, 0.7, 0.9)	(0.7, 0.9, 1.0)	(0.3, 0.5, 0.7)
A2	(7, 9, 10)	(9, 10, 10)	(5, 7, 9)	(5, 7, 9)	(0.7, 0.9, 1.0)	(0.9, 1.0, 1.0)	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)
A3	(9, 10, 10)	(9, 10, 10)	(5, 7, 9)	(3, 5, 7)	(0.9, 1.0, 1.0)	(0.9, 1.0, 1.0)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)
A4	(3, 5, 7)	(5, 7, 9)	(5, 7, 9)	(7, 9, 10)	(0.3, 0.5, 0.7)	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)	(0.7, 0.9, 1.0)
A5	(3, 5, 7)	(7, 9, 10)	(9, 10, 10)	(5, 7, 9)	(0.3, 0.5, 0.7)	(0.7, 0.9, 1.0)	(0.9, 1.0, 1.0)	(0.5, 0.7, 0.9)
A6	(5, 7, 9)	(3, 5, 7)	(9, 10, 10)	(3, 5, 7)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.9, 1.0, 1.0)	(0.3, 0.5, 0.7)
A7	(7, 9, 10)	(7, 9, 10)	(7, 9, 10)	(5, 7, 9)	(0.7, 0.9, 1.0)	(0.7, 0.9, 1.0)	(0.7, 0.9, 1.0)	(0.5, 0.7, 0.9)
A8	(3, 5, 7)	(5, 7, 9)	(3, 5, 7)	(7, 9, 10)	(0.3, 0.5, 0.7)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.7, 0.9, 1.0)
A9	(5, 7, 9)	(5, 7, 9)	(3, 5, 7)	(9, 10, 10)	(0.5, 0.7, 0.9)	(0.5, 0.7, 0.9)	(0.3, 0.5, 0.7)	(0.9, 1.0, 1.0)
A10	(7, 9, 10)	(7, 9, 10)	(3, 5, 7)	(7, 9, 10)	(0.7, 0.9, 1.0)	(0.7, 0.9, 1.0)	(0.3, 0.5, 0.7)	(0.7, 0.9, 1.0)

Table 9.12 The advantage and disadvantage of the assets

Assets	Advantage				Disadvantage			
	C_1	C_2	C_3	C_4	C_1	C_2	C_3	C_4
A1	2.95	0.4	0.4	0	0.2	2.15	3.3	4.4
A2	2.95	3.75	1.8	1.2	0.2	0	1.2	1.6
A3	4.75	3.75	1.8	0	0	0	1.2	4.4
A4	0	0.4	1.8	3.3	4.75	2.15	1.2	0.2
A5	0	2.15	0	1.2	4.75	0.4	4.9	1.6
A6	1.2	0	0	0	1.95	5.75	4.9	4.4
A7	2.95	2.15	0.4	1.2	0.2	0.4	3.3	1.6
A8	0	0.4	4.6	3.3	4.75	2.15	0	0.2
A9	1.2	0.4	4.6	5.1	1.95	2.15	0	0
A10	2.95	2.15	4.6	3.3	0.2	0.4	0	0.2

It may be noted that the FP score of the asset A6 is 0. The asset A6 is least preferable to the given investor in comparison to the other assets on the criteria C_2, C_3 and C_4 (see the ratings in the columns 3, 4 and 5 of the Table 9.11). By considering investor preferences for the asset A6 on all the

Table 9.13 The strength and weakness of the assets

Assets	Fuzzy strength	Fuzzy weakness	Strength using fuzzy strength & fuzzy weakness	Weakness using fuzzy strength & fuzzy weakness
A1	(2.115, 2.825, 3.455)	(5.665, 7.46, 8.71)	19.375	109.75
A2	(6.47, 8.035, 9.045)	(1.42, 2.02, 2.5)	126.0475	7.92
A3	(7.01, 8.695, 9.825)	(2.16, 3.28, 4.28)	116.445	10.32
A4	(2.61, 3.67, 4.51)	(5.21, 6.655, 7.765)	28.75	87.875
A5	(2.295, 2.75, 2.99)	(6.645, 8.935, 10.695)	13.025	135.15
A6	(0.6, 0.84, 1.08)	(10.9, 13.725, 15.485)	0	252.375
A7	(4.05, 5.175, 6.045)	(3.25, 4.31, 5)	62.1575	44.2825
A8	(4.57, 6.19, 7.31)	(4.37, 5.575, 6.565)	58.6575	47.7825
A9	(5.71, 7.93, 9.65)	(2.91, 3.515, 3.905)	101.65	14.775
A10	(7.62, 10.005, 11.715)	(0.52, 0.64, 0.72)	184.125	0

Table 9.14 FP scores of the assets

Assets	FP scores	Normalized scores
A1	0.1500	0.0280
A2	0.9409	0.1758
A3	0.9186	0.1716
A4	0.2465	0.0461
A5	0.0879	0.0164
A6	0	0
A7	0.5840	0.1091
A8	0.5511	0.1030
A9	0.8731	0.1631
A10	1.0000	0.1868

four criteria, its fuzzy strength turns out to be the least and its fuzzy weakness becomes the highest in comparison to the other assets (see columns 2 and 3 of the Table 9.13).

9.4.3 Asset Allocation

This stage concerns the choice of combination of the assets to build portfolios that manage the trade-off between financial goal and ethical goal corresponding to investor preferences.

We use each asset's normalized FP score (see Table 9.14), EP score (see Table 9.7), $\beta = 0.1$, $\gamma = 1.5$ to construct the portfolio selection problem P(9.1).

$$\begin{aligned}
\max\ Z(x) &= 0.0280x_1 + 0.1758x_2 + 0.1716x_3 + 0.0461x_4 + 0.0164x_5 \\
&\quad +0x_6 + 0.1091x_7 + 0.1030x8 + 0.1631x_9 + 0.1868x_{10} \\
\text{subject to}& \\
&0.0792x_1 + 0.1104x_2 + 0.2313x_3 + 0.0297x_4 + 0.1117x_5 + 0.0621x_6 \\
&\quad +0.0899x_7 + 0.0483x8 + 0.0606x_9 + 0.1766x_{10} \geq 0.1\,, \\
&-x_1\ \ln x_1 - x_2\ \ln x_2 - x_3\ \ln x_3 - x_4\ \ln x_4 - x_5\ \ln x_5 - x_6\ \ln x_6 \\
&\quad -x_7\ \ln x_7 - x_8\ \ln x_8 - x_9\ \ln x_9 - x_{10}\ \ln x_{10} \geq 1.5\,, \\
&x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 1\,, \\
&x_i \geq 0\,, \quad i = 1, \ldots, 10\,.
\end{aligned}$$

The portfolio selection strategy is obtained by solving the above model using LINGO 12.0. The corresponding computational results are presented in Table 9.15. We can see from the results that the capital is allocated comparatively more to the assets A2, A3, A9 and A10 whose FP scores are high and EP scores are within acceptable limits. Also, the portfolio is comparatively dispersed since all the assets received some proportion of the capital.

Table 9.15 The proportions of the assets in the obtained portfolio

	Allocation									
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
Portfolio	0.00108	0.24061	0.20635	0.0021	0.00071	0.00039	0.02098	0.01678	0.1512	0.3598

Table 9.16 Portfolio selection corresponding to the different preset entropy values at $\beta = 0.1$

		Allocation									
Entropy γ	Financial goal	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
0.50000	0.18507	0	0.09019	0.03805	0	0	0	0	0	0.00664	0.86512
1.00000	0.18175	0	0.19069	0.11995	0	0	0	0.00012	0.00006	0.04693	0.64225
1.50000	0.17374	0.00108	0.24061	0.20635	0.0021	0.00071	0.00039	0.02098	0.01678	0.1512	0.3598
2.00000	0.15044	0.02573	0.19087	0.1803	0.03288	0.02198	0.0176	0.07726	0.07113	0.16068	0.22157
2.30258	0.10021	0.09967	0.10035	0.10033	0.09975	0.09962	0.09954	0.10004	0.10001	0.10029	0.1004

Table 9.17 Portfolio selection corresponding to different preset ethical levels at different entropy levels

			Allocation									
Entropy γ	Ethicality β	Financial goal	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10
0.5												
	0.18	0.18474	0	0.03522	0.10765	0	0	0	0	0	0.00135	0.85578
	0.19	0.18308	0	0	0.24497	0	0	0	0	0	0	0.75503
	0.20	0.18030	0	0	0.42779	0	0	0	0	0	0	0.57221
	0.22	0.17474	0	0	0.79342	0	0	0	0	0	0	0.20658
1.0												
	0.18	0.18100	0	0.11893	0.25677	0	0	0	0.00024	0.00009	0.02362	0.60035
	0.19	0.17948	0	0.08732	0.38809	0	0	0	0.00057	0.0002	0.01701	0.50681
	0.20	0.17614	0.00036	0.06962	0.56787	0.00026	0.00046	0.00011	0.00577	0.00221	0.01857	0.33477
	0.206	0.16625	0.01071	0.03336	0.70728	0.00319	0.02375	0.0065	0.01698	0.00583	0.00917	0.18323
1.5												
	0.15	0.17374	0.00108	0.24041	0.20670	0.00210	0.00071	0.00039	0.02099	0.01678	0.15098	0.35986
	0.16	0.17318	0.00199	0.20139	0.27640	0.00288	0.00158	0.00080	0.02449	0.01746	0.11390	0.35912
	0.17	0.17097	0.00475	0.15613	0.36432	0.00481	0.00474	0.00225	0.03071	0.01944	0.08019	0.33266
	0.18	0.16426	0.01432	0.09940	0.47676	0.00892	0.01959	0.00854	0.03806	0.02053	0.04491	0.26897
2.0												
	0.12	0.15044	0.02573	0.19087	0.18030	0.03288	0.02198	0.01760	0.07726	0.07113	0.16068	0.22157
	0.13	0.15042	0.02620	0.18758	0.18768	0.03253	0.02282	0.01796	0.07694	0.06962	0.15497	0.22370
	0.14	0.14843	0.03225	0.15694	0.24708	0.03146	0.03286	0.02282	0.07485	0.05944	0.11295	0.22935
	0.15	0.13556	0.05502	0.09526	0.33315	0.03378	0.07583	0.04414	0.06955	0.04470	0.05565	0.19292

If the investor is not satisfied with diversification of the obtained portfolio, more portfolios can be generated by varying the preset entropy value γ in the above problem, see the computational results listed in Table 9.16. It can be seen from the results that when the preset entropy value is increased, the allocation of the capital becomes more diverse. When the preset entropy value is 0.5, the capital is only allocated to the four assets, i.e., $A2$, $A3$, $A9$ and $A10$; when the preset entropy value is 2.30258, the capital is almost uniformly allocated to all the 10 assets. This implies that a higher entropy threshold will ensure a more diversified investment. However, accompanied with more diversified investment, the achievement level of the financial goal becomes smaller, i.e., falls from 0.18507 to 0.10021, which is in line with risk-return trade off.

Further, in order to understand the repercussions of the compromise between financial and ethical criteria on investment decision, we present sensitivity analysis w.r.t. changes in the β in the above problem. The computational results are listed in Table 9.17. Note that while investors seek to maximize the overall financial goal, they also want to be sure of an acceptable level of ethicality of the portfolio as well. However, subject to a given level of entropy (thereby portfolio diversification) it can be achieved only up to a particular level of ethicality after which the portfolio selection problem becomes infeasible. Thus, the investor can realize the desired level of ethicality by choosing an appropriate level of diversification of the portfolio. In general, higher the entropy, lower the ethical score of portfolio and vice versa.

9.5 Comments

In this chapter, we have presented the following facts:

- A nonlinear optimization model based on the inputs from Fuzzy-MCDM and Fuzzy-AHP techniques has been introduced to attain the convergence of ethicality and financial optimality in portfolio selection.
- Fuzzy-MCDM has been used for determining the overall financial quality score of each asset with respect to four key financial criteria: short term return, long term return, risk and liquidity.
- Fuzzy-AHP has been used to measure the EP score of each asset with respect to the ethical criteria.
- A nonlinear optimization model has been used to obtain portfolios that reflect investor preferences for financial quality and the desired level of ethicality in portfolio construction.

- Portfolio diversification as captured in the entropy constraint function has been used as the pivot for attaining the convergence of the twin considerations under the contemplation of the investors.
- The computational results based on real-world data have been provided to demonstrate the effectiveness of the proposed methodology for ethicality and financial optimality issues in portfolio selection.
- The main advantage of the portfolio selection approach is that if the investor is not satisfied with the financial quality of the portfolio obtained, more portfolios can be generated by varying the preset entropy threshold in the optimization model. Likewise, if the investor is not satisfied with the ethicality of the portfolio obtained, more portfolios can be generated by varying the preset ethical threshold in the model.

Chapter 10
Multi-criteria Portfolio Optimization Using Support Vector Machines and Genetic Algorithms

Abstract. Given that not all the assets available in the market are appropriate for a given investor, it is desirable to stratify these assets into different classes on the basis of some predefined characteristics. Furthermore, using investor preferences, one needs to select some good quality assets from a given class to build an optimal portfolio. The focus of this chapter is to present a hybrid approach to portfolio selection using investor preferences in terms of selection of assets from a particular class that suits the given investor-type. The support vector machine (SVM) with radial basis function kernel is used to classify the assets into three classes. The optimal portfolio selection is achieved using a model that is based on four financial criteria: short term return, long term return, risk, and liquidity. A real coded genetic algorithm (RCGA) is designed to solve the portfolio selection model.

10.1 Overview of Support Vector Machines

SVMs is a machine-learning technique based on statistical learning theory. An important property that made SVMs a promising tool is their implementation of structural risk minimization which aims to minimize a bound on the generalization error rather than on the empirical error. SVMs attempts to construct an optimal separating hyperplane by transforming a nonlinear object into a high dimension feature space and thus gives a good generalization performance on a wide range of problems, such as text categorization [67], pattern recognition [99] and bioinformatics [122]. The details on SVMs can be found in [14, 22]. The SVMs have been successfully used in several financial applications such as stock selection, credit rating, time series prediction, insurance claim fraud detection, corporate credit rate prediction and bankruptcy prediction, see [31, 52, 69, 80, 111, 114, 117].

We briefly describe the basic SVMs concepts for typical two-class classification problems. In SVMs technique, the main aim of an SVM classifier is to determine the decision boundary or hyperplane that optimally separates two

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classes of input data points. Consider a training set of instance-label pairs $(x_i, y_i), i = 1, 2, \ldots, m$ where $x_i \in R^n$ and $y_i \in \{-1, +1\}$. Suppose we have a separating hyperplane $w.x + b = 0$ that separates the positive from the negative examples, i.e., the two classes. Here,

- w is the normal vector of the hyperplane and b is the bias value.
- $\frac{|b|}{\| w \|}$ is perpendicular distance from the hyperplane to the origin and $\| w \|$ is the Euclidean norm of w.

Let $d_+(d_-)$ be the shortest distance from the separating hyperplane to the closest positive (negative) example called support vectors. The 'margin' of a separating hyperplane is defined to be $d_+ + d_-$. Thus, the margin is the width that the boundary could have before hitting a data point. With an aim of minimizing the chances of misclassification, the support vector algorithm looks for the separating hyperplane with largest margin; thus, resulting in orientation of the separating hyperplane in such a way as to be as far as possible from the closest members of both the classes.

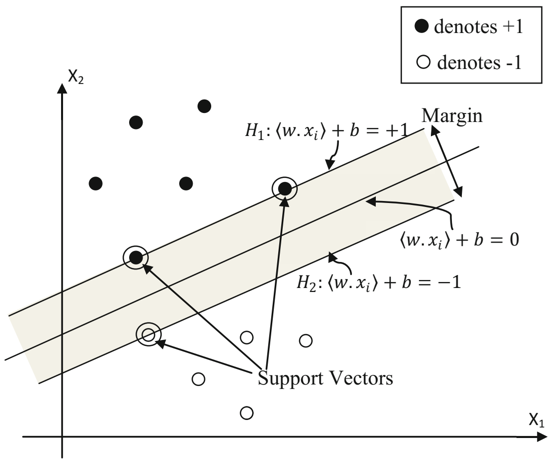

Fig. 10.1 Linear separating hyperplanes for the separable case

Referring to Fig. 10.1, it can be seen that a SVM is implemented by choosing a scale for the variables w and b so that the training data can be described by

$$\langle w.x_i \rangle + b \geq +1 \quad \text{for } y_i = +1\,, \tag{10.1}$$

$$\langle w.x_i \rangle + b \leq -1 \quad \text{for } y_i = -1\,. \tag{10.2}$$

The inequalities (10.1) and (10.2) can be combined into the following set of inequalities:

$$y_i(\langle w.x_i\rangle + b) - 1 \geq 0, \quad i = 1, 2, \ldots, m. \tag{10.3}$$

If we now just consider the points that lie closest to the separating hyperplane, i.e., the support vectors (shown in circles in Figure 10.1), then the two planes H_1 and H_2 on which these points lie can be described by

$$\langle w.x_i\rangle + b = +1 \quad \text{for}\, H_1\,, \tag{10.4}$$

$$\langle w.x_i\rangle + b = -1 \quad \text{for}\, H_2\,. \tag{10.5}$$

For a hyperplane which is equidistant from H_1 and H_2, $d_+ = d_- = \dfrac{1}{\| w \|}$ and the margin $d_+ + d_- = \dfrac{2}{\| w \|}$. The SVM finds an optimal separating hyperplane with the maximum margin by solving the following quadratic optimization problem:

$$\begin{aligned} &\min_{w,b} \frac{1}{2} w^T.w \\ &\text{subject to} \\ &y_i(\langle w.x_i\rangle + b) - 1 \geq 0, \quad i = 1, 2, \ldots, m. \end{aligned} \tag{10.6}$$

The above problem can be solved by solving the corresponding Lagrangian. Let the Lagrange multipliers $\alpha \in R^m$, where $\alpha_i \geq 0, i = 1, 2, \ldots, m$ be chosen to formulate the Lagrangian

$$L_P(w, b, \alpha) = \frac{1}{2} w^T.w - \sum_{i=1}^{m} \alpha_i (y_i(\langle w.x_i\rangle + b) - 1). \tag{10.7}$$

We must now minimize L_P with respect to w and b. This is equivalent to solve the 'dual' problem: maximize L_D subject $\nabla_{w,b} L_p(w, b, \alpha) = 0$ and $\alpha_i \geq 0, i = 1, 2, \ldots, m$. This particular dual formulation of the problem is called the Wolfe dual [34]. We have $\nabla_{w,b} L_p(w, b, \alpha) = 0$ implies

$$\frac{\partial L_P}{\partial w} = 0 \Rightarrow w = \sum_{i=1}^{m} \alpha_i y_i x_i\,, \tag{10.8}$$

$$\frac{\partial L_P}{\partial b} = 0 \Rightarrow \sum_{i=1}^{m} \alpha_i y_i = 0\,. \tag{10.9}$$

Substituting (10.8) and (10.9) in (10.7), the following dual formulation is obtained

$$\max_{\alpha} L_D(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2}\sum_{i,j=1}^{m} \alpha_i\alpha_j y_i y_j \langle x_i.x_j \rangle$$

subject to

$$\alpha_i \geq 0, \quad i = 1, \ldots, m, \tag{10.10}$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0. \tag{10.11}$$

To find the optimal hyperplane, the dual Lagrangian $L_D(\alpha)$ must be maximized with respect to α. This is a quadratic optimization problem that can be solved by using some standard optimization method. The solution $\alpha_i, i = 1, 2, \ldots, m$ for the dual optimization problem determines the parameters w^* and b^* of the optimal hyperplane. The following optimal decision hyperplane $f(x, \alpha^*, b^*)$ is obtained along with an indicator decision function $sign[f(x, \alpha^*, b^*)]$.

$$f(x, \alpha^*, b^*) = \sum_{i=1}^{m} y_i \alpha_i^* \langle x_i.x \rangle + b^* . \tag{10.12}$$

The above concepts can also be extended to the non-separable case (see Figure 10.2). The goal is to construct a hyperplane that makes the smallest number of errors. For this purpose, we introduce the non-negative slack variables $\xi_i \geq 0$, $i = 1, \ldots, m$ such that

$$\langle w.x_i \rangle + b \geq +1 - \xi_i \quad \text{for } y_i = +1 ,$$
$$\langle w.x_i \rangle + b \leq -1 + \xi_i \quad \text{for } y_i = -1 .$$

If errors happen to the classification of training data then ξ_i will be larger than zero. Thus, a lower $\sum_{i=1}^{m} \xi_i$ is preferred when determining the separating hyperplane. For this purpose, a penalty parameter $C > 0$ is added to control the allowable error ξ_i and thus the new quadratic optimization problem is obtained as

$$\min_{w,b,\xi} \frac{1}{2} w^T.w + C\sum_{i=1}^{m} \xi_i$$

subject to

$$y_i(\langle w.x_i \rangle + b) + \xi_i - 1 \geq 0, \quad \xi_i \geq 0, i = 1, \ldots, m. \tag{10.13}$$

To solve the above model, we again take recourse to duality theorem. Thus, we maximize the Lagrangian $L_D(\alpha)$ as in the separable case,

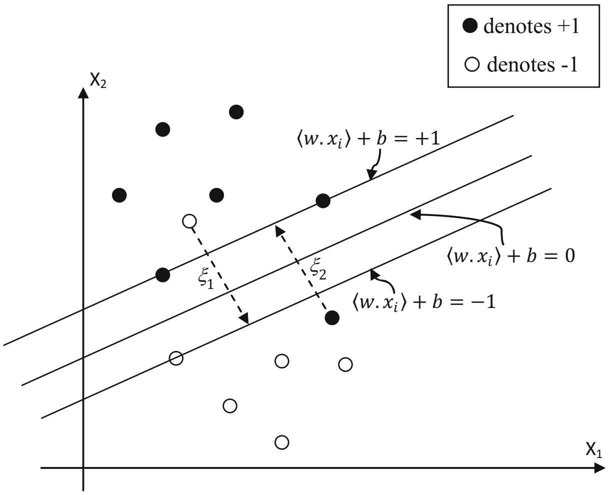

Fig. 10.2 Linear separating hyperplanes for the non-separable case

$$\max_{\alpha} L_D(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j \langle x_i . x_j \rangle$$

subject to

$$0 \le \alpha_i \le C, \quad i = 1, \ldots, m, \tag{10.14}$$

$$\sum_{i=1}^{m} \alpha_i y_i = 0. \tag{10.15}$$

The penalty parameter C, which is now the upper bound on $\alpha_i, i = 1, 2, \ldots, m$, is determined by the user. Finally, the optimal decision hyperplane is obtained as in (10.12).

When a linear boundary is inappropriate, the nonlinear SVM can map the input vector into a high dimensional feature space via a mapping function Φ, which is also called kernel function. In the dual Lagrange (see (10.10)-(10.11)), the inner products are replaced by the kernel function as follows:

$$\langle \Phi(x_i) . \Phi(x_j) \rangle = K(x_i, x_j).$$

Now, the nonlinear SVM dual Lagrangian $L_D(\alpha)$ is obtained as follows:

$$\max_{\alpha} L_D(\alpha) = \sum_{i=1}^{m} \alpha_i - \frac{1}{2} \sum_{i,j=1}^{m} \alpha_i \alpha_j y_i y_j K(x_i, x_j)$$

subject to

Constraints (10.14) – (10.15).

The above optimization model can be solved using the method for solving the optimization model in the separable case. Now, the optimal hyperplane has the following form:

$$\begin{aligned} f(x, \alpha^*, b^*) &= \sum_{i=1}^{m} y_i \alpha_i^* \langle \Phi(x_i).\Phi(x) \rangle + b^* \\ &= \sum_{i=1}^{m} y_i \alpha_i^* K(x_i, x) + b^*. \end{aligned}$$

Depending upon the choice of kernel, the bias b can form an implicit part of the kernel function. Therefore, if a bias term can be accommodated within the kernel function then the nonlinear support vector classifier can be obtained as

$$f(x, \alpha^*, b^*) = \sum_{i=1}^{m} y_i \alpha_i^* \langle \Phi(x_i).\Phi(x) \rangle = \sum_{i=1}^{m} y_i \alpha_i^* K(x_i, x).$$

The detailed discussion on kernel functions can be found in [14]. Some commonly used kernel functions include polynomial, radial basis function (RBF) and sigmoid kernel, which are given as under.

Polynomial kernel:

$$K(x_i, x_j) = (1 + x_i.x_j)^d$$

RBF kernel:

$$K(x_i, x_j) = \exp(-\gamma \parallel x_i - x_j \parallel^2)$$

Sigmoid kernel:

$$K(x_i, x_j) = \tanh(k x_i.x_j - \delta)$$

Note that the classification accuracy improves if the kernel parameters in the above kernel functions are chosen properly.

10.2 Multiobjective Portfolio Selection Model

In this section, we formulate portfolio selection problem as an optimization problem with multiple objectives assuming that the investor allocate his/her wealth among n assets that offer random rates of return. We introduce some notation as follows:

10.2.1 Notation

r_i: the expected rate of return of the i-th asset,

x_i: the proportion of the total funds invested in the i-th asset,

y_i: a binary variable indicating whether the i-th asset is contained in the portfolio, where

$$y_i = \begin{cases} 1, & \text{if } i\text{-th asset is contained in the portfolio}, \\ 0, & \text{otherwise}, \end{cases}$$

r_i^{12}: the average performance of the i-th asset during a 12-month period,

r_i^{36}: the average performance of the i-th asset during a 36-month period,

r_{it}: the historical return of the i-th asset over the past period t,

u_i: the maximal fraction of the capital allocated to the i-th asset,

l_i: the minimal fraction of the capital allocated to the i-th asset,

L_i: the turnover rate of the i-th asset,

h: the number of assets held in the portfolio,

T: the total time span.

We consider the following objective functions and constraints in the multiobjective portfolio selection problem.

10.2.2 Objective Functions

Short Term Return
The short term return of the portfolio is expressed as

$$f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i,$$

where $r_i^{12} = \frac{1}{12}\sum_{t=1}^{12} r_{it}$, $i = 1, 2, \ldots, n$; r_{it} is determined from the historical data.

Long Term Return
The long term return of the portfolio is expressed as

$$f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i,$$

where $r_i^{36} = \frac{1}{36}\sum_{t=1}^{36} r_{it}$, $i = 1, 2, \ldots, n$.

Risk
The portfolio risk using semi-absolute deviation measure is expressed as

$$f_3(x) = w(x) = \frac{1}{T}\sum_{t=1}^{T} w_t(x) = \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2T} .$$

Liquidity
The portfolio liquidity is expressed as

$$f_4(x) = \sum_{i=1}^{n} L_i x_i .$$

10.2.3 Constraints

Capital budget constraint on the assets is expressed as

$$\sum_{i=1}^{n} x_i = 1 .$$

Maximal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \leq u_i y_i , \quad i = 1, 2, \ldots, n .$$

Minimal fraction of the capital that can be invested in a single asset is expressed as

$$x_i \geq l_i y_i , \quad i = 1, 2, \ldots, n .$$

Number of assets held in the portfolio is expressed as

$$\sum_{i=1}^{n} y_i = h .$$

No short selling of assets is expressed as

$$x_i \geq 0 , \quad i = 1, 2, \ldots, n .$$

10.2.4 The Decision Problem

The multiobjective mixed integer nonlinear programming problem for portfolio selection is formulated as follows:

$$\textbf{P(10.1)}\quad \max\ f_1(x) = \sum_{i=1}^{n} r_i^{12} x_i$$

$$\max\ f_2(x) = \sum_{i=1}^{n} r_i^{36} x_i$$

$$\min\ f_3(x) = w(x) = \sum_{t=1}^{T} \frac{\left|\sum_{i=1}^{n}(r_{it} - r_i)x_i\right| + \sum_{i=1}^{n}(r_i - r_{it})x_i}{2T}$$

$$\max\ f_4(x) = \sum_{i=1}^{n} L_i x_i$$

subject to

$$\sum_{i=1}^{n} x_i = 1\,, \tag{10.16}$$

$$\sum_{i=1}^{n} y_i = h\,, \tag{10.17}$$

$$x_i \le u_i y_i\,, \quad i = 1, 2, \ldots, n\,, \tag{10.18}$$

$$x_i \ge l_i y_i\,, \quad i = 1, 2, \ldots, n\,, \tag{10.19}$$

$$x_i \ge 0\,, \quad i = 1, 2, \ldots, n\,, \tag{10.20}$$

$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, n\,. \tag{10.21}$$

10.3 Numerical Illustration

Presented hereunder are the results of an empirical study for which we have relied on a data set of daily closing prices in respect of 150 assets listed on the NSE, Mumbai, India.

10.3.1 Asset Classes

We use the following three classes of assets as discussed in Section 7.5.1 of chapter 7.

(i) ***Class 1: Liquid assets***
Assets in class 1 are categorized as liquid assets since mean value for liquidity is the highest in this class. This class is typified by low but widely varying returns.

(ii) ***Class 2: High-yield assets***
Assets in class 2 are categorized as high-yield ones since they have rather high returns. On the expected lines of risk-return relationship, these assets also show high standard deviation. Although, investors may profit from the high returns, they also have to endure the high risk. These assets have low liquidity amongst all the three classes indicating that high-yielding investment involves a longer time horizon.

(iii) ***Class 3: Less-risky assets***
Assets in class 3 are categorized as less-risky assets since compared to other classes, these assets have the lowest standard deviation for the class. The return is not high but medium. The liquidity is medium too.

10.3.2 Classification of Assets Using SVM

We use LIBSVM software [16] to perform multiclass SVM experiments. To allow for multiclass classification, LIBSVM uses one-against-one approach, see [16]. We split the data into two subsets: a training set of 60%(data of 90 assets) and a testing set of 40%(data of 60 assets) of the total data (data of 150 assets), respectively. Consider three evaluation indices to perform classification, namely, asset returns (the average 36-month performance of the assets), standard deviation and liquidity. Out of all available kernels for SVM, the advantage of using the linear kernel SVM is that there are no parameters to tune except for constant C, but it affects the prediction performance for the cases where the training data is not separable by a linear SVM [25]. For the nonlinear SVM, there is an additional parameter, the kernel parameter, to tune. As discussed in Section 10.1, there are three commonly kernel functions for nonlinear SVM, namely, the RBF, the polynomial and the sigmoid kernel. The RBF kernel nonlinearly maps the samples into a higher dimension space unlike the linear kernel, so it can handle the case when the relation between class labels and attributes is nonlinear. Furthermore, the linear kernel is a special case of RBF. In addition, the sigmoid kernel behaves like RBF for certain parameters, however, it is not valid under some parameters. The polynomial kernel takes a longer time in the training stage of SVM and it is reported to provide worse results than the RBF kernel, see [52, 114]. We, therefore, use the RBF kernel SVM as the default model. There are two parameters associated with the RBF kernel, C and γ. It is not known beforehand which values of C and γ are the best for one problem; consequently, some kind of model selection (parameter search) approach must be employed. We conduct a gridsearch to find the best values of C and γ using 10-fold cross validation.

10-fold cross validation is a technique used to test how well a model adapts to fresh, previously unseen data. The procedure for 10-fold cross validation is as follows:

Step 1: Use a random sampling procedure to split the entire training set into 10 sub-samples. Lets call these samples $S_1, S_2, \ldots, S_{10}$.
Step 2: As a first step, remove sample S_{10} from the training set.
Step 3: Train machine learning algorithm using data from samples S_1 to S_9.
Step 4: Once the machine has built a model based on data from samples S_1 to S_9, it sees how accuratly the model predicts the unseen data of S_{10}. Error rates are stored by the system.
Step 5: Once the accuracy of predicting the values in S_{10} is tested, put S_{10} back into the training set.
Step 6: Repeat Steps 2 to 5 by removing samples $S_1, S_2, \ldots, S_9$ one at a time.

At the end of the sequence, the 10 results from the folds can be averaged to produce a single estimation of the model's predictive potential. A big advantage of the 10-fold cross validation method is that all observations are used for both training and validation, and each observation is used for validation exactly once. This leads to a more accurate way to measure how efficiently the algorithm has 'learned' a concept, based on training data set.

The SVM experiments are conducted with different pairs of (C,γ) and the one with the best cross validation accuracy is selected. Note that cross validation procedure can prevent the overfitting problem. It is well established that trying exponentially growing sequences of C and γ is a practical method to identify good parameters, for example, $C = 2^{22}, 2^{23}, \ldots, 2^{28}, \gamma = 2^{-5}, 2^{-3}, 2^{-1}, 2^{1}$. After conducting the grid-search on the training data, we find that the optimal (C,γ) is $(2^{25}, 2^{-3}\ )$ with the cross-validation rate of 97.7528%(see Figure 10.3). Table 10.1 summarizes the results of the grid-search. After obtaining the optimal (C,γ), the SVM classifier is built for the training data. The testing data is then input to the SVM classifier and the prediction accuracy is found to be 91.6666%. The classification confusion matrix containing information about actual and predicted classifications done by the obtained SVM classifier, is presented in Table 10.2.

The 21 financial assets classified in class 1, 20 financial assets classified in class 2 and 19 financial assets classified in class 3 comprise the population for the three classes. We construct a portfolio comprising 7 assets with the corresponding upper and lower bounds of capital budget allocation. Table 10.3 provides the input data corresponding to the short term return, long term return, risk and liquidity of assets in classes 1, 2 and 3.

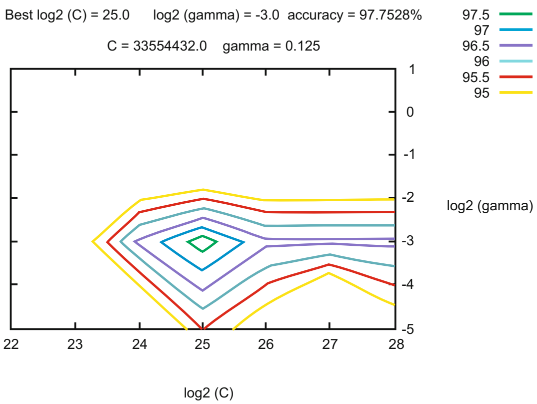

Fig. 10.3 Grid-search using $C = 2^{22}, 2^{23}, \ldots, 2^{28}, \gamma = 2^{-5}, 2^{-3}, 2^{-1}, 2^{1}$

Table 10.1 The results of grid-search

C	γ			
	2^{-5}	2^{-3}	2^{-1}	2^{1}
2^{22}	93.2584	94.382	93.2584	89.8876
2^{23}	94.382	94.382	93.2584	89.8876
2^{24}	93.2584	96.6292	93.2584	89.8876
2^{25}	95.5056	**97.7528**	93.2584	89.8876
2^{26}	94.382	96.6292	93.2584	89.8876
2^{27}	92.1348	96.6292	93.2584	89.8876
2^{28}	94.382	96.6292	93.2584	89.8876

Table 10.2 Classification confusion matrix for test data

Actual	Predicted		
	Class 1	Class 2	Class 3
Class 1	20	1	2
Class 2	1	19	1
Class 3	0	0	16
Total	21	20	19

Table 10.3 Input data for SVM classified assets

Class	Criteria	Assets										
		A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11
Class 1												
	Short term return	-0.06077	0.17239	0.02755	0.05613	0.06638	0.06897	0.18109	0.05893	-0.00300	0.15257	0.14955
	Long term return	0.11213	0.15058	0.11744	0.10497	0.11670	0.105482	0.20361	0.17278	0.09370	0.11504	0.10050
	Risk	0.19770	0.18976	0.11776	0.18504	0.21634	0.16645	0.28852	0.23321	0.19970	0.12004	0.20528
	Liquidity	0.00610	0.00359	0.00309	0.00569	0.00936	0.01679	0.01149	0.00633	0.00184	0.00243	0.00570
Class 2												
	Short term return	0.07506	0.17499	0.15611	0.20095	0.07138	0.33979	0.27290	0.07716	0.14934	0.11277	0.15108
	Long term return	0.1645	0.19277	0.21365	0.21710	0.19819	0.40086	0.30831	0.17448	0.23202	0.27857	0.27476
	Risk	0.31853	0.14009	0.16853	0.20634	0.27252	0.42326	0.38341	0.11526	0.13323	0.36504	0.33879
	Liquidity	0.00489	0.00031	0.00125	0.00123	0.00133	0.00351	0.00240	0.00104	0.00234	0.00631	0.00414
Class 3												
	Short term return	0.11486	0.09949	0.00614	0.14239	0.09360	0.09156	0.11308	0.15842	0.15477	0.10980	0.10272
	Long term return	0.149209	0.13586	0.16622	0.14868	0.14382	0.16847	0.14905	0.10491	0.19491	0.15926	0.12071
	Risk	0.08682	0.15059	0.10149	0.09806	0.07715	0.13906	0.07572	0.16118	0.10979	0.06117	0.07586
	Liquidity	0.00088	0.00503	0.00201	0.00251	0.00079	0.00470	0.00058	0.00547	0.00266	0.00132	0.00412

Class	Criteria	Assets									
		A12	A13	A14	A15	A16	A17	A18	A19	A20	A21
Class 1											
	Short term return	0.14446	0.05044	0.09697	0.00650	0.18838	0.004889	0.07629	0.15397	0.01637	-0.03562
	Long term return	0.15531	0.15601	0.08175	0.10379	0.14987	0.09224	0.10131	0.22522	0.12500	0.14412
	Risk	0.30816	0.25208	0.29367	0.18687	0.21569	0.18920	0.10462	0.36056	0.13280	0.19432
	Liquidity	0.04798	0.00938	0.01743	0.00224	0.01596	0.01449	0.013335	0.02748	0.00266	0.00080
Class 2											
	Short term return	0.06771	0.19231	0.19926	0.30955	0.17552	0.16299	0.08523	0.11344	0.15496	-
	Long term return	0.17132	0.14649	0.17984	0.34733	0.29632	0.18133	0.22948	0.17929	0.36700	-
	Risk	0.20750	0.35137	0.13248	0.42247	0.23701	0.13970	0.31436	0.16257	0.29759	-
	Liquidity	0.00179	0.00115	0.00059	0.00104	0.00084	0.00008	0.00046	0.00043	0.00042	-
Class 3											
	short term return	0.12789	0.14324	0.09095	0.09319	0.08036	0.18872	0.17020	0.02793	-	-
	Long term return	0.12385	0.13537	0.19162	0.13501	0.13470	0.28212	0.25382	0.13964	-	-
	Risk	0.08101	0.08893	0.06740	0.09570	0.07977	0.44632	0.51691	0.11507	-	-
	Liquidity	0.00085	0.00037	0.00040	0.00303	0.00201	0.01378	0.01216	0.00066	-	-

10.3.3 Real Coded GA to Solve Portfolio Selection Model

To construct optimum portfolio for a given investor, one needs to pick the assets from the suitable class as per investor preferences and find the best combination of assets according to the portfolio selection model P(10.1) discussed in Section 10.2.4.

Genetic algorithms (GAs) were proposed by Holland [51] and since then have been well developed and documented in the literature. The financial application of GA is growing with successful applications in trading system [21, 24], portfolio selection [19, 76, 83], bankruptcy prediction [70], credit evaluation [118] and budget allocation [98]. Here, we use RCGA. The basic difference between the conventional GA and RCGA is how the encoding of chromosomes is performed. Coding of the variables is essential for an efficient GA. The RCGA, which uses real numbers for encoding, converges more quickly toward optima than GA. It also overcomes the difficulty of the Hamming cliff, which may arise in the conventional GA when the Hamming distance (which is defined in binary coding in conventional GA) between two adjacent integers (in decimal code) is very large. In such cases, a large number of bits must be altered to change an integer to the adjacent one, which causes a reduction in the efficiency of conventional GA. An overview of conventional GA is given below.

To solve a problem with GA, an encoding mechanism must first be designed to represent each solution as a chromosome. A fitness function is then defined to measure the goodness of a chromosome. The GA searches the solution space using a population, which is a set of chromosomes at each generation. During each generation, the three genetic operators, namely, selection, crossover and mutation are applied to the population several times to form a new population. The selection operation forms a parent population that is used for creating the next generation. Given a crossover rate, the crossover operation recombines the two selected chromosomes to form offspring. Given a mutation rate, the mutation operation randomly alters selected positions in a selected chromosome. The new population is then generated by replacing some chromosomes in the parent population with the offspring. This process is repeated until some termination condition, e.g., a maximum number of generations, is reached.

Following are the details of RCGA used to solve the model P(10.1).

- **Chromosome encoding**

A gene in a chromosome is characterized by two factors: Locus (i.e., the position of the gene located within the structure of chromosome) and allele (i.e., the value the gene takes). In the proposed encoding method, the length of the chromosome is taken to be n, same as the number of available assets in the class for which optimal portfolio is being obtained. Let the solution

$x = (x_1, x_2, \ldots, x_n)$ be represented by the chromosome Ch_k which is encoded as an array, as follows:

$$Ch_k = X_k[i] = x_i, i = 1, 2, \ldots, n, k = 1, 2, \ldots, \ popsize.$$

Here, *popsize* defines the number of chromosomes initialized to constitute population of one generation. In this encoding method, the position of the gene x_i, for $i = 1, 2, \ldots, n$, is used to represent the ID number of the asset and the gene's value is used to represent the corresponding proportion of the total funds invested in the i-th asset. The initialization algorithm to create first generation of chromosomes of size *popsize* is as follows:

Step 1: For $k = 1$ to *popsize*, repeat Step 2 to Step 4.
Step 2: Randomly select h assets out of the n available assets for initialization to satisfy the cardinality constraint (10.17).
Step 3: For $i = 1$ to n, repeat Step 4.
Step 4: If the i-th asset has been selected in Step 2, then assign $y_i = 1$ and randomly generate $x_i \in [l_i y_i, u_i y_i]$, i.e., $x_i \in [l_i, u_i]$; otherwise, assign $y_i = 0$ and hence $x_i = 0$. Thus, $X_k[i] = x_i$. This step ensures the constraints (10.18)-(10.21) are satisfied.

• **Fitness evaluation**
The fitness evaluation function must consider all the desired objective functions and make rational trade-offs among them. The only constraint of the model P(10.1) that is not incorporated in the chromosome design is the capital budget constraint (10.16). To design the fitness function, this constraint is incorporated in the RCGA process by assigning a penalty P to the infeasible chromosomes. The penalty parameter is used to apply sufficient selective pressure on the fitness function to avoid infeasible chromosomes. In the case of no violation of the capital budget constraint, the penalty parameter P will be zero; it is positive otherwise. It may be noted that if the penalty is too high or too low, then the problem might become very difficult for the GA to solve. At the beginning of the search process, a large penalty discourages the exploration of the infeasible region. Conversely, if the penalty is too low, significant search time (generations) will be spent exploring the infeasible region because the penalty will be negligible compared with the objective function(s). It has been observed in the literature that penalties that are functions of the distance from feasibility (the completion cost) perform better. The selection of appropriate penalty is vital for faster convergence and more precision. However, it is difficult to determine an appropriate penalty parameter that is problem specific. We use a static penalty in which the penalty parameter remains constant during the entire RCGA process. Let

$$f_5(x) = \left| \sum_{i=1}^{n} x_i - 1 \right|.$$

The penalty levied on the infeasible chromosomes is

$$P = \begin{cases} 10^4 * f_5(x), & \text{if } f_5(x) > 10^{-3}, \\ 0, & \text{otherwise.} \end{cases}$$

Since the model P(10.1) is a multiobjective programming problem, we use weighted sum approach to combine multiple objective functions into a single composite function. Thus, the resulting fitness function fit_k corresponding to chromosome Ch_k, where $k = 1, 2, \ldots, popsize$, is defined as weighted sum of the objective functions of model P(10.1) with a penalty parameter for infeasible chromosomes as follows:

$$fit_k = w_1 f_1(x) + w_2 f_2(x) - w_3 f_3(x) + w_4 f_4(x) - P\,,$$

where $w_j > 0,\ j = 1, 2, 3, 4$ such that $\sum_{j=1}^{4} w_j = 1$, is the weight given to the j-th objective function, highlighting the relative importance of a particular objective in a given class. The objective is now to find the solution chromosome Ch_k corresponding to the optimum (maximum value) of the fitness function fit_k.

- **Elitism**

To preserve and use the previously determined best solution in subsequent generations, an elite-preserving operator is often used. In addition to an overall increase in performance, there is another advantage of using elitism. In an elitist GA, the statistics of the population of best solutions cannot degrade with generations. The elite count (t) indicates the number of individuals that are guaranteed to be included in the next generation without the selection, crossover and mutation operations being performed. We use $t = 1$ to retain the fittest individual of the current population when constructing the population for the next generation.

- **Selection**

The selection method determines how chromosomes are selected from the current population to be considered parents for the crossover operation. The goal of the selection(reproduction) operator is to choose individuals that, on average, are more fit than others to pass their genes to the next generation. We employ 4-player tournament selection as a selection mechanism. Four individuals are randomly selected and the individual with the highest fitness is selected for the parent population. Recall that we already have one member of the next generation as a result of performing the elitism operation. Let Ch'_k, where $k = 2, 3, \ldots, popsize$, constitute the parent population that will yield the remaining $popsize - 1$ members of the next generation after the crossover and mutation operations are performed. The remaining $popsize - 1$ chromosomes for the parent population are generated using the 4-player tournament selection as follows:

Step 1: For $k = 2$ to *popsize*, repeat Step 2 to Step 5.
Step 2: Randomly generate four integers $sel_1, sel_2, sel_3, sel_4 \in [1, popsize]$. These represents the selection of chromosomes $Ch_{sel_1}, Ch_{sel_2}, Ch_{sel_3}, Ch_{sel_4}$ for 4-player tournament selection.
Step 3: If $fit_{sel_1} \geq fit_{sel_2}$, let $m = sel_1$; otherwise, $m = sel_2$.
Step 4: If $fit_{sel_3} \geq fit_{sel_4}$, let $t = sel_3$; otherwise, $t = sel_4$.
Step 5: If $fit_m \geq fit_t$, let $Ch'_k = Ch_m$; otherwise, $Ch'_k = Ch_t$.

• **Crossover operator**
The two parent chromosomes, if selected for mating pool, reproduce two child chromosomes (offspring) using the crossover operation. The crossover probability $p_c \in (0, 1)$ represents the chance that the two selected chromosomes will crossover. For each potential crossover, a random number between 0 and 1 is generated. If the number of selected chromosomes is odd, then the above procedure is repeated until one more chromosome is selected or the number of selected chromosomes becomes even. Standard crossover operators have a high probability of violating the cardinality constraint (10.17) of the model P(10.1). Thus, we use the shrinking crossover (SX) operator [19]. SX revises the two-point crossover by moving the second crossover point leftward until there are an equal number of selected assets between and including the two crossover points for both the selected parents and then exchanging the gene values of the parent chromosomes to produce offspring. The algorithm of the SX operation is given below:

Step 1: For $k = 2$ to *popsize*, repeat Step 2.
Step 2: Randomly generate a real number r from the interval $(0, 1)$. The chromosome Ch'_k is selected as a parent for crossover if $r < p_c$.
Step 3: Denote the selected parents as $S_1, S_2, \ldots$ and divide them into the following pairs: $(S_1, S_2), (S_3, S_4), \ldots$.
Step 4: For each pair of selected parents, e.g., (S_1, S_2), randomly select two positions $a, b \in [1, n]$. If $a < b$ then $pos_1 = a$ and $pos_2 = b$; otherwise, $pos_1 = b$ and $pos_2 = a$.
Step 5: Until S_1 & S_2 have an equal number of selected assets between and including pos_1 & pos_2, repeat Step 6.
Step 6: $pos_2 = pos_2 - 1$.
Step 7: If $pos_1 = pos_2$ go to Step 4; otherwise, go to Step 8.
Step 8: For $i = pos_1$ to pos_2, repeat Step 9 to Step 11.
Step 9: $temp = S_1[i]$.
Step 10: $S_1[i] = S_2[i]$.
Step 11: $S_2[i] = temp$.

Figure 10.4 depicts the SX operation between two randomly generated positions, pos_1=14 and pos_2=18, for the selected parents. Here, pos_1=14 corresponds to asset A14 having proportions of the total funds x_{14}=0.1463 and

x_{14}=0.1231 for the first and the second parents, respectively. pos_2=18 corresponds to asset A18 having proportions of the total funds x_{18}=0.2749 and x_{18}=0 for the first and the second parents, respectively. Because these two positions do not have the same number of selected assets for both the parents, pos_2 is shifted left. Finally, two-point crossover occurs between pos_1=14 and pos_2=17 when the number of selected assets between and including the two positions stated above matches for both the parents.

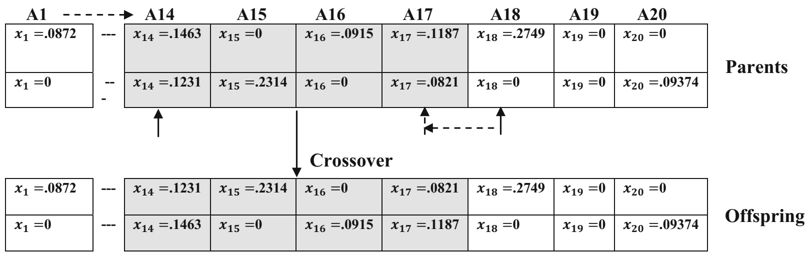

Fig. 10.4 Shrinking crossover operation

It may be noted that even after each crossover operation, the feasibility of the chromosomes is retained. Note that all the parents that are not selected for the crossover operation will be retained in the population for the next generation.

• **Mutation operator**

Out of several available mutation operators, we use a variant of swap mutation to ensure that the constraints (10.17)-(10.21) of model P(10.1) are not violated. Given some probability of mutation $p_m \in (0, 1)$, a chromosome is selected for the process of mutation. The mutation process is summarized as follows:

Step 1: Set $k = 2$.

Step 2: If $k <= popsize$, go to Step 3; otherwise, stop.

Step 3: Randomly generate a real number r from the interval $(0, 1)$. If $r < p_m$, select the chromosome Ch'_k for mutation and go to Step 4; otherwise, $k = k + 1$ and go to Step 2.

Step 4: Randomly select two positions, $pos_1, pos_2 \in [1, n]$.

(a) If $y_{pos_1} = 0$ and $y_{pos_2} = 0$, $k = k + 1$ and go to Step 2.

(b) If $y_{pos_1} = 0$ and $y_{pos_2} = 1$ then

a. Swap the information about asset allocation, i.e., $y_{pos_1} = 1$ and $y_{pos_2} = 0$.

b. $x_{pos_2} = 0$ corresponding to $y_{pos_2} = 0$.
c. Regenerate $x_{pos_1} \in [l_{pos_1} y_{pos_1}, u_{pos_1} y_{pos_1}]$, i.e., $x_{pos_1} \in [l_{pos_1}, u_{pos_1}]$ corresponding to $y_{pos_1} = 1$.

(c) If $y_{pos_1} = 1$ and $y_{pos_2} = 0$ then
a. Swap the information about asset allocation, i.e., $y_{pos_1} = 0$ and $y_{pos_2} = 1$.
b. $x_{pos_1} = 0$ corresponding to $y_{pos_1} = 0$.
c. Regenerate $x_{pos_2} \in [l_{pos_2} y_{pos_2}, u_{pos_2} y_{pos_2}]$, i.e., $x_{pos_2} \in [l_{pos_2}, u_{pos_2}]$ corresponding to $y_{pos_2} = 1$.

(d) If $y_{pos_1} = 1$ and $y_{pos_2} = 1$ then regenerate $x_{pos_1} \in [l_{pos_1} y_{pos_1}, u_{pos_1} y_{pos_1}]$, i.e., $x_{pos_1} \in [l_{pos_1}, u_{pos_1}]$ and $x_{pos_2} \in [l_{pos_2} y_{pos_2}, u_{pos_2} y_{pos_2}]$, i.e., $x_{pos_2} \in [l_{pos_2}, u_{pos_2}]$.

Figure 10.5 depicts the process of modified swap mutation in a selected chromosome with pos_1=14 and pos_2=16. Here, in the selected chromosome, pos_1=14 corresponds to asset A14 having a proportion of the total funds x_{14}=0.1231 and pos_2=16 corresponds to asset A16 having a proportion of the total funds x_{16}=0. This implies that corresponding to the two selected positions $y_{14} = 1$ and $y_{16} = 0$. After the information is swapped about the asset allocation, we get y_{14}=0 and y_{16}=1. Thus, corresponding to y_{14}=0 we set x_{14}=0 and corresponding to y_{16}=1, we randomly regenerate $x_{14} \in [l_{14}, u_{14}]$ (for illustration purpose $x_{14} \in [0.08, 0.3]$) that takes a value x_{14}=0.1753.

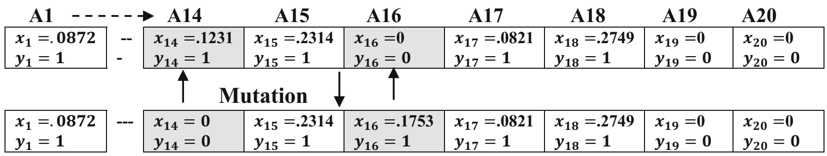

Fig. 10.5 A variant of swap mutation operation

10.3.4 Asset Allocation

In what follows, we present experimental results. The RCGA is coded in C++ on a personal computer with 2.8 GHz Intel Core2Duo CPU and a 4 GB RAM. The primary attributes of the problem instances solved are summarized in Table 10.4.

- **Class 1**

Using the input parameters from Tables 10.3 and 10.4 corresponding to class 1, the multiobjective portfolio selection model is formulated as follows:

Table 10.4 Primary attributes of the problem instances solved

	Class 1	Class 2	Class 3
Length of a chromosome (Number of assets)	21	20	19
Number of assets to be selected (h)	7	7	7
$u_i, \forall i$	0.3	0.3	0.3
$l_i, \forall i$	0.08	0.08	0.08
w_1	0.2	0.3	0.17
w_2	0.25	0.35	0.23
w_3	0.2	0.2	0.45
w_4	0.35	0.15	0.15

$$
\begin{aligned}
\max f_1(x) = &-0.06077x_1 + 0.17239x_2 + 0.02755x_3 + 0.05613x_4 + 0.06638x_5 \\
&+0.06897x_6 + 0.18109x_7 + 0.05893x_8 - 0.00300x_9 + 0.15257x_{10} \\
&+0.14955x_{11} + 0.14446x_{12} + 0.05044x_{13} + 0.09697x_{14} + 0.00650x_{15} \\
&+0.18838x_{16} + 0.00488x_{17} + 0.07629x_{18} + 0.15397x_{19} \\
&+0.01637x_{20} - 0.03562x_{21} \\
\max f_2(x) = &\ 0.11213x_1 + 0.15058x_2 + 0.11744x_3 + 0.10497x_4 + 0.11670x_5 \\
&+0.10548x_6 + 0.20361x_7 + 0.17278x_8 + 0.09370x_9 + 0.11504x_{10} \\
&+0.10050x_{11} + 0.15531x_{12} + 0.15601x_{13} + 0.08175x_{14} \\
&+0.10379x_{15} + 0.14987x_{16} + 0.09224x_{17} + 0.10131x_{18} \\
&+0.22522x_{19} + 0.12500x_{20} + 0.14412x_{21} \\
\min f_3(x) = &\ (6.46119x_1 + 6.20732x_2 + 4.90974x_3 + 5.806900x_4 + 6.29314x_5 \\
&+5.87325x_6 + 6.90162x_7 + 6.97952x_8 + 6.12055x_9 + 4.94361x_{10} \\
&+5.73353x_{11} + 7.30630x_{12} + 6.07934x_{13} + 7.24465x_{14} \\
&+5.74783x_{15} + 6.09968x_{16} + 6.00977x_{17} + 4.46672x_{18} \\
&+7.87304x_{19} + 5.00413x_{20} + 6.34961x_{21})/36
\end{aligned}
$$

$$
\begin{aligned}
\max f_4(x) = {} & 0.00610x_1 + 0.00359x_2 + 0.00309x_3 + 0.00569x_4 + 0.00936x_5 \\
& +0.01679x_6 + 0.01149x_7 + 0.00633x_8 + 0.00184x_9 + 0.00243x_{10} \\
& +0.00570x_{11} + 0.04798x_{12} + 0.00938x_{13} + 0.01743x_{14} \\
& +0.00224x_{15} + 0.01596x_{16} + 0.01449x_{17} + 0.01333x_{18} \\
& +0.02748x_{19} + 0.00266x_{20} + 0.00080x_{21}
\end{aligned}
$$

subject to

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} \\
& +x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} + x_{21} = 1\,, \\
& y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} \\
& +y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} + y_{21} = 7\,, \\
& x_i \le 0.3y_i\,, \quad i = 1, 2, \ldots, 21\,, \\
& x_i \ge 0.08y_i\,, \quad i = 1, 2, \ldots, 21\,, \\
& x_i \ge 0\,, \quad i = 1, 2, \ldots, 21\,, \\
& y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 21\,.
\end{aligned}
$$

Using $w_1 = 0.2$, $w_2 = 0.25$, $w_3 = 0.2$ and $w_4 = 0.35$, we obtain the desired portfolio by employing the RCGA. Note that class 1 is of liquid assets, therefore, the highest weightage (w_4) is given to liquidity objective.

• **Class 2**

Using the input parameters from Tables 10.3 and 10.4 corresponding to class 2, the multiobjective portfolio selection model is formulated as follows:

$$
\begin{aligned}
\max f_1(x) = {} & 0.07506x_1 + 0.17499x_2 + 0.15611x_3 + 0.20095x_4 + 0.07138x_5 \\
& +0.33979x_6 + 0.27290x_7 + 0.07716x_8 + 0.14934x_9 + 0.11277x_{10} \\
& +0.15108x_{11} + 0.06771x_{12} + 0.19231x_{13} + 0.19926x_{14} \\
& +0.30955x_{15} + 0.17552x_{16} + 0.16299x_{17} + 0.08523x_{18} \\
& +0.11344x_{19} + 0.15496x_{20} \\
\max f_2(x) = {} & 0.16456x_1 + 0.19277x_2 + 0.21365x_3 + 0.21710x_4 + 0.19819x_5 \\
& +0.40086x_6 + 0.30831x_7 + 0.17448x_8 + 0.23202x_9 + 0.27857x_{10} \\
& +0.27476x_{11} + 0.17132x_{12} + 0.14649x_{13} + 0.17984x_{14} \\
& +0.34733x_{15} + 0.29632x_{16} + 0.18133x_{17} + 0.229487x_{18} \\
& +0.17929x_{19} + 0.36700x_{20} \\
\min f_3(x) = {} & (7.61839x_1 + 4.76400x_2 + 5.70491x_3 + 6.23081x_4 + 6.40035x_5 \\
& +9.54210x_6 + 8.36534x_7 + 4.39545x_8 + 5.07153x_9 + 7.85012x_{10} \\
& +7.07354x_{11} + 6.24403x_{12} + 7.71357x_{13} + 4.66028x_{14} \\
& +8.92113x_{15} + 6.16019x_{16} + 5.19704x_{17} + 7.45273x_{18} \\
& +5.52108x_{19} + 7.62776x_{20})/36
\end{aligned}
$$

$$
\begin{aligned}
\max f_4(x) = {} & 0.00489x_1 + 0.00031x_2 + 0.00125x_3 + 0.00123x_4 + 0.00133x_5 \\
& +0.00351x_6 + 0.00240x_7 + 0.00104x_8 + 0.00234x_9 + 0.00631x_{10} \\
& +0.00414x_{11} + 0.00179x_{12} + 0.00115x_{13} + 0.00059x_{14} \\
& +0.00104x_{15} + 0.00084x_{16} + 0.00008x_{17} + 0.00046x_{18} \\
& +0.00043x_{19} + 0.00042x_{20}
\end{aligned}
$$

subject to

$$
\begin{aligned}
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13} \\
& +x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} + x_{20} = 1\,, \\
& y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13} \\
& +y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} + y_{20} = 7\,, \\
& x_i \le 0.3y_i\,, \quad i = 1,2,\ldots,20\,, \\
& x_i \ge 0.08y_i\,, \quad i = 1,2,\ldots,20\,, \\
& x_i \ge 0\,, \quad i = 1,2,\ldots,20\,, \\
& y_i \in \{0,1\}\,, \quad i = 1,2,\ldots,20\,.
\end{aligned}
$$

Using $w_1 = 0.3$, $w_2 = 0.35$, $w_3 = 0.2$ and $w_4 = 0.15$, we obtain the desired portfolio by employing the RCGA. Note that class 2 is of high-yield assets, therefore, the highest weightages (w_1 and w_2) are given to return objectives.

- **Class 3**

Using the input parameters from Tables 10.3 and 10.4 corresponding to class 3, the multiobjective portfolio selection model is formulated as follows:

$$
\begin{aligned}
\max f_1(x) = {} & 0.09949x_1 + 0.00614x_2 + 0.14239x_3 + 0.09360x_4 + 0.09156x_5 \\
& +0.11308x_6 + 0.15842x_7 + 0.15477x_8 + 0.10980x_9 + 0.10272x_{10} \\
& +0.11486x_{11} + 0.12789x_{12} + 0.14324x_{13} + 0.09095x_{14} + 0.09319x_{15} \\
& +0.08036x_{16} + 0.18872x_{17} + 0.17020x_{18} + 0.02793x_{19} \\
\max f_2(x) = {} & 0.13586x_1 + 0.16622x_2 + 0.14868x_3 + 0.14382x_4 + 0.16847x_5 \\
& +0.14905x_6 + 0.10491x_7 + 0.19491x_8 + 0.15926x_9 + 0.14920x_{10} \\
& +0.12071x_{11} + 0.12385x_{12} + 0.13537x_{13} + 0.19162x_{14} + 0.13501x_{15} \\
& +0.13470x_{16} + 0.28212x_{17} + 0.25382x_{18} + 0.13964x_{19} \\
\min f_3(x) = {} & (4.19795x_1 + 5.74022x_2 + 4.443904x_3 + 4.60141x_4 + 3.61455x_5 \\
& +5.45973x_6 + 3.82595x_7 + 5.41154x_8 + 4.78747x_9 + 3.42547x_{10} \\
& +4.05060x_{11} + 4.03797x_{12} + 3.99987x_{13} + 3.90462x_{14} + 4.35063x_{15} \\
& +4.30645x_{16} + 8.40087x_{17} + 10.29254x_{18} + 4.85847x_{19})/36
\end{aligned}
$$

$$\max f_4(x) = 0.00412x_1 + 0.00088x_2 + 0.0050x_3 + 0.00201x_4 + 0.00251x_5$$
$$+0.00079x_6 + 0.00470x_7 + 0.00058x_8 + 0.00547x_9 + 0.00266x_{10}$$
$$+0.00132x_{11} + 0.00085x_{12} + 0.00037x_{13} + 0.00040x_{14} + 0.00303x_{15}$$
$$+0.00201x_{16} + 0.01378x_{17} + 0.01216x_{18} + 0.00066x_{19}$$

subject to

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} + x_{11} + x_{12} + x_{13}$$
$$+x_{14} + x_{15} + x_{16} + x_{17} + x_{18} + x_{19} = 1\,,$$
$$y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7 + y_8 + y_9 + y_{10} + y_{11} + y_{12} + y_{13}$$
$$+y_{14} + y_{15} + y_{16} + y_{17} + y_{18} + y_{19} = 7\,,$$
$$x_i \le 0.3y_i\,, \quad i = 1, 2, \ldots, 19\,,$$
$$x_i \ge 0.08y_i\,, \quad i = 1, 2, \ldots, 19\,,$$
$$x_i \ge 0\,, \quad i = 1, 2, \ldots, 19\,,$$
$$y_i \in \{0, 1\}\,, \quad i = 1, 2, \ldots, 19\,.$$

Using $w_1 = 0.17$, $w_2 = 0.23$, $w_3 = 0.45$ and $w_4 = 0.15$, we obtain the desired portfolio by employing the RCGA. Note that class 3 is of less-risky assets, therefore, the highest weightage (w_3) is given to risk objective.

We perform certain experiments to choose best parameter settings of the RCGA parameters. We solve the model P(10.1) using different values of the RCGA parameters. The parameters used and corresponding computational results are given in Table 10.5. To compare the results, we use the relative error (RE) index which is defined as follows:

$$RE = (\text{Maximal fitness - Actual fitness})/\text{Maximal fitness} \times 100\%$$

where the maximal fitness is the maximum of the fitness values for all the computational results obtained.

From Table 10.5, we see that the relative error corresponding to each setting of parameters does not exceed 2%, which demonstrate that the proposed RCGA is effective at setting the parameters. We use the

Table 10.5 Results corresponding to different settings of the RCGA parameters

p_c	p_m	*popsize*	Generation runs	Fitness	Relative error (%)
0.3	0.05	50	3000	0.045824	1.5702
0.3	0.06	100	5000	0.046243	0.6702
0.35	0.06	50	3000	0.046045	1.0955
0.35	0.07	100	5000	0.045776	1.6733
0.4	0.07	50	5000	0.045906	1.3941
0.45	0.06	50	3000	0.04568	1.8795
0.45	**0.07**	**100**	**5000**	**0.046555**	**0**
0.5	0.07	50	3000	0.046434	0.2599

parameter settings from Table 10.5 that correspond to the maximum fitness, i.e., $popsize = 100, p_c = 0.45, p_m = 0.07,$ generation runs $= 5000$ to run the RCGA to obtain the optimal portfolio for the three classes. To further demonstrate the efficiency of the algorithm, we performed 20 runs of RCGA to check the stability of the solutions obtained. Solution statistics for the 20 runs are reported in Table 10.6. Figures 10.6-10.8 shows the sensitivity of the maximum fitness attained in each RCGA run for all the three classes. The best solution out of 20 RCGA runs has been reported in Table 10.7. Table 10.8 present proportions of the assets in the obtained portfolios. A comparison of the solutions for the three classes listed in Table 10.7 highlights that if investors are looking for high liquidity, they should invest in class 1 assets, i.e., liquid assets. The attainment level of liquidity of the portfolio build from class 1 assets is higher in comparison to class 2 and class 3, but it is accompanied by a medium risk level. If investors are looking for returns, they should invest in class 2 assets, i.e., high-yield assets. The attainment level of returns of the portfolio build from class 2 is higher in comparison to class 1 and class 3, but that carries a higher risk level too. If investors are looking for safe investment, they should invest in class 3 assets, i.e., less-risky assets. The attainment level of risk of the portfolio build from class 3 is lower in comparison to class 1 and class 2, but that supposes accepting medium level of expected returns.

Table 10.6 Solution statistics for 20 RCGA runs for the various classes

	Class		
	Class 1	Class 2	Class 3
Best Fitness	0.046555	0.153982	0.029875
Average Fitness	0.0463282	0.1532298	0.0297486
Standard Deviation	0.000105604	0.000439727	6.52028E-05
Coefficient of Variation(%)	0.227947185	0.286972182	0.219179529

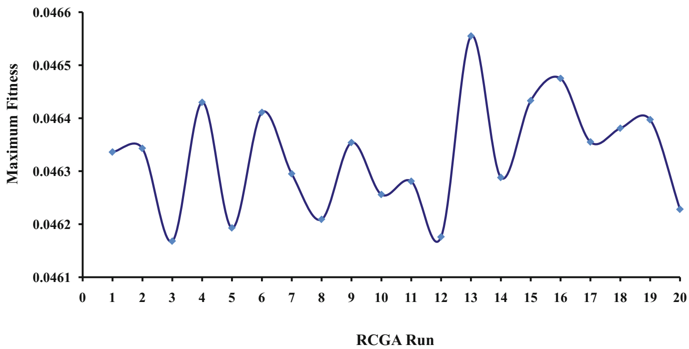

Fig. 10.6 Maximum fitness vs. RCGA run for class 1

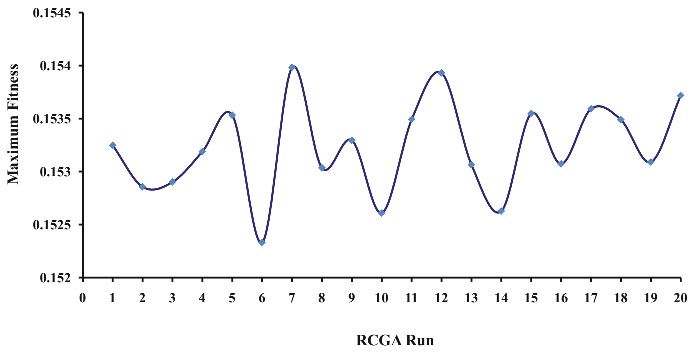

Fig. 10.7 Maximum fitness vs. RCGA run for class 2

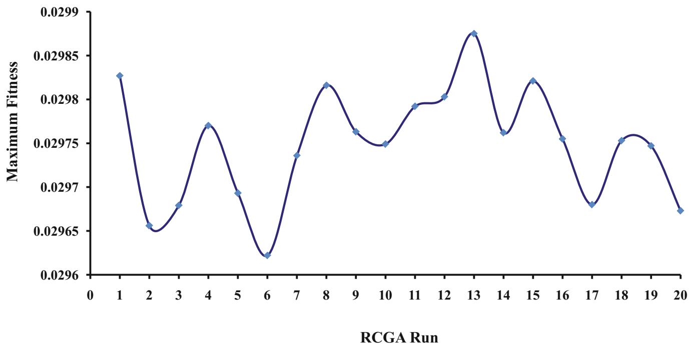

Fig. 10.8 Maximum fitness vs. RCGA run for class 3

Table 10.7 Attainment values of the various objectives

Objective	Class		
	Class 1	Class 2	Class 3
Short term return	0.165134	**0.269611**	0.152307
Long term return	0.181297	**0.340519**	0.202851
Risk	0.190471	0.232001	**0.158369**
Liquidity	**0.017995**	0.002114	0.006025

Table 10.8 The proportions of the assets in the obtained portfolios

Class	Allocation										
	A1	A2	A3	A4	A5	A6	A7	A8	A9	A10	A11
Class 1	0	0.081128	0	0	0	0	0.292279	0	0	0.080134	0.081262
Class 2	0	0	0	0.083699	0	0.297952	0.087654	0	0	0	0.084653
Class 3	0	0	0.080994	0	0.083948	0	0.080611	0.282336	0	0	0
	A12	A13	A14	A15	A16	A17	A18	A19	A20	A21	
	0.090762	0	0	0	0.085479	0	0	0.289822	0	0	
	0	0	0	0.28384	0.082484	0	0	0	0.080638	-	
	0	0.081813	0.09132	0	0	0.299872	0	0	-	-	

10.4 Comments

In this chapter, we have presented the following facts:

- SVMs has been introduced to categorize the financial assets into three predefined classes, based on three financial evaluation indices, namely, return, risk and liquidity.
- RCGA has been used to solve the multiobjective portfolio selection problem considering short term return, long term return, risk and liquidity.
- The main advantage of the SVM classifier is that once such a classifier is obtained, it can then be used to classify any set of randomly chosen assets into the relevant classes. This provide investors a *prima facie* information about the class of the assets and thus help investors to decide the appropriate investment alternatives. The investors, then, may pick and choose from among these alternatives by obtaining the desired portfolio with the help of RCGA.
- The advantage of using RCGA is that we need not linearize the risk objective and can solve portfolio selection model P(10.1) in its original form.
- Using the computational results, it has been shown that the approach discussed here is capable of classifying assets with good accuracy and is also capable of yielding optimal portfolios for each class of assets based on the investor preferences regarding the financial criteria used.

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